SOME TWO-GENERATOR ONE-RELATOR NON-HOPFIAN GROUPS

BY GILBERT BAUMSLAG[1] AND DONALD SOLITAR

Communicated by Lipman Bers, January 8, 1962

In 1951 Graham Higman claimed (in [1]) that every finitely generated group with a single defining relation is Hopfian,[2] attributing this fact to B. H. Neumann and Hanna Neumann. However we shall show that this is not, in any way, the case. For example the group

$$G = gp(a, b;\, a^{-1}b^2a = b^3) \tag{1}$$

is non-Hopfian. Hence the following question of B. H. Neumann [2, p. 545] has a *negative* answer: *Is every two-generator non-Hopfian group infinitely related?*

This group G turns out to be useful for deciding a somewhat different kind of question. For Graham Higman[3] has pointed out that G can, of course, be generated by a and b^4. However it transpires that in terms of these generators G requires *more than one relation* to define it. Thus Higman has produced a counter-example to the following well-known conjecture: *Let G be generated by n elements $a_1, a_2, \cdots, a_n$ and let r be the least number in any set of defining relations between $a_1, a_2, \cdots, a_n$. Then $n-r$ is an invariant of G (i.e. does not depend on the particular basis $a_1, a_2, \cdots, a_n$).* This conjecture has received some attention in the past; indeed there is a "proof" of it by Petresco [3].

The group defined by (1) is clearly only one of a larger family of groups of the kind

$$G = gp(a, b;\, a^{-1}b^la = b^m). \tag{2}$$

It is convenient at this point to introduce a definition. Thus we say two nonzero integers l and m are meshed if either

(i) l or m divides the other,

or,

(ii) l and m have precisely the same prime divisors. This definition enables us to distinguish easily between the Hopfian and the non-Hopfian groups in the family of groups (2). For the following theorem holds.

THEOREM 1. *Let l and m be nonzero integers. Then*

[1] Supported by Grant G 19674 from the National Science Foundation.

[2] A group G is Hopfian if $G/N \cong G$ implies $N=1$; otherwise G is non-Hopfian.

[3] In a letter.

$$G = gp(a, b; a^{-1}b^l a = b^m)$$

is Hopfian if and only if l and m are meshed.

The proof of Theorem 1 is in three parts. Thus we prove

(a) if l or m divides the other, then G is residually finite[4] and therefore Hopfian (Mal'cev [4]);

(b) if l and m are meshed but neither divides the other, then every ependomorphism of G is an automorphism and so G is Hopfian;

(c) if l and m are not meshed, then G is non-Hopfian.

It is perhaps worthwhile to sketch the proof of (c). Here we may assume, without loss of generality, the existence of a prime p dividing l but not m. Hence the mapping

$$\eta: a \to a, \qquad b \to b^p$$

defines an ependomorphism of G. Now it follows from the work of Magnus [5; 6] that

$$[b^{l/p}, a]^p b^{l-m} \neq 1.$$

However

$$([b^{l/p}, a]^p b^{l-m})\eta = [b^l, a]^p b^{p(l-m)} = 1.$$

Therefore the kernel K of η is *nontrivial* and as

$$G(=G\eta) \cong G/K$$

we have proved G is non-Hopfian.

The following theorem is a direct consequence of Theorem 1. It illustrates strikingly that hopficity is a finiteness condition of the weakest kind.

THEOREM 2. *The group*

$$G = gp(a, b; a^{-1}b^{12}a = b^{18})$$

is Hopfian but possesses a normal subgroup of finite index which is non-Hopfian.

It turns out that G'', the second derived group of

$$G = gp(a, b; a^{-1}b^2a = b^3)$$

is free. This fact enables us to prove the following theorem (cf. B H. Neumann [2, p. 544]).

[4] G is residually finite if for each $x \in G$ ($x \neq 1$) there corresponds a normal subgroup $N_x(G)$ such that G/N_x is finite and $x \notin N_x$.

THEOREM 3. *The groups*

$$G = gp(a, b; a^{-1}b^2a = b^3)$$

and

$$H = gp(c, d; c^{-1}d^2c = d^3, ([c, d]^2c^{-1})^2 = 1)$$

are homomorphic images of each other; however they are not isomorphic.

Finally we employ Theorem 1 to provide the first instance of a two-generator group which is soluble-of-length-three and *non-Hopfian.* Thus

THEOREM 4. *There exists a two-generator group which is soluble-of-length-three and non-Hopfian.*

Theorem 4 may be compared with the results of B. H. Neumann and Hanna Neumann [7] and P. Hall [8].

REFERENCES

1. Graham Higman, *A finitely related group with an isomorphic proper factor group.* J. London Math. Soc. **26** (1951), 59–61.

2. B. H. Neumann, *An essay on free products of groups with amalgamations*, Philos. Trans. Roy. Soc. London. Ser. A. **246** (1954), 503–554.

3. J. Petresco, *Systèmes minimaux de relations fondamentales dans les groupes de rang fini*, Séminaire Paul Dubreil et Charles Pisot, 9e année: 1955/56.

4. A. I. Mal'cev, *On isomorphic representations of infinite groups by matrices*, Mat. Sb. **8** (1940), 405–422.

5. ———, *Über diskontinuierliche Gruppen mit einer definierenden Relation* (*Der Freiheitssatz*), J. Reine Angew. Math. **163** (1930), 141–165.

6. W. Magnus, *Das Identitätsproblem für Gruppen mit einer definierenden Relation*, Math. Ann. **106** (1932), 295–307.

7. B. H. Neumann, and Hanna Neumann, *Embedding theorems for groups*, J. London Math. Soc. **34** (1959), 465–479.

8. P. Hall, *The Frattini subgroups of finitely generated groups*, Proc. London Math. Soc. **11** (1961), 327–352.

NEW YORK UNIVERSITY AND
ADELPHI COLLEGE