THE MINIMUM MODULUS OF INTEGRAL FUNCTIONS OF SMALL ORDER[1]

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Let $f(z)$ be an integral function and

$$M(r) = \max_{|z|=r} |f(z)|, \qquad m(r) = \min_{|z|=r} |f(z)|.$$

If $f(z)$ has order 0, then the $\cos \pi\rho$-theorem [1, p. 40] implies that, for any $\epsilon > 0$, there is a sequence of $r \to \infty$ on which

$$\log m(r) > (1 - \epsilon) \log M(r). \tag{1}$$

If $\log M(r) = O(\log^2 r)$ $(r \to \infty)$ then Hayman [2] proved that (1) holds outside a set of finite logarithmic measure. If $f(z)$ has order at most ρ and C is any constant, then Kjellberg [3, p. 20] showed that

$$\text{lower log-dens } E\{m(r) > C\} \geqq 1 - 2\rho,$$

and that $1 - 2\rho$ is best-possible. Surveys of the main results of this kind are given in [1; 3].

In this field I have proved a number of results, of which I state the following. For a given function $\psi(r)$ $(r > 0)$ I use the notation

$$\psi_1(r) = d\psi(r)/d \log r, \qquad \psi_2(r) = d^2\psi(r)/d \log^2 r,$$

when these derivatives exist.

THEOREM 1. *Suppose that*

$$\log M(r) \leqq \{1 + o(1)\}\psi(r) \qquad (r \to \infty),$$

that $\psi_1(r) = o\{\psi(r)\}$ $(r \to \infty)$ *and that for* $r \geqq r_0$, $1/\psi_2(r)$ *is positive, monotonic decreasing, and a convex function of* $\log r$. *Then if* $0 < \delta < 1$ *and* $\epsilon > 0$ *we have*

$$\text{lower log-dens } E\left\{\log \frac{m(r)}{M(r)} > -(1 + \epsilon)\delta^{-1}(2 - \delta)\frac{1}{2}\pi^2\psi_2(r)\right\} \geqq 1 - \delta.$$

We call a function $L(x)$ continuous and positive for $x \geqq 0$, slowly oscillating, if for each fixed $\lambda > 0$,

[1] This, in the main, is an abstract of a thesis submitted for the degree of Ph.D. in the University of London, under the supervision of Professor W. K. Hayman.

$$\lim_{x\to\infty} L(\lambda x)/L(x) = 1.$$

THEOREM 2. *If* $\psi_2(r)$ *is slowly oscillating,* $\epsilon>0$, *and*

$$\log M(r) = o\{\psi(r)\} \qquad (r\to\infty),$$

then on a sequence of $r\to\infty$,

$$\log m(r) > \left\{1 - \frac{1}{2}(\pi^2+\epsilon)\psi_2(r)/\psi(r)\right\} \log M(r).$$

THEOREM 3. *If* $\epsilon>0$ *and*

$$\log\log M(r) = o\{(\log r)^{1/2}\} \qquad (r\to\infty)$$

then

$$\text{log-dens } E\,\{\log m(r) > (1-\epsilon)\log M(r)\} = 1.$$

THEOREM 4. *If* $f(z)$ *is transcendental and has order at most* ρ, *and* C *is any constant, then*

$$\text{lower log-dens } E\,\{\log m(r) > C\log r\} \geqq 1 - 2\rho.$$

The proof of Theorem 2 is based on the ideas of Beurling's theorem [1, p. 4] as used by Kjellberg [3, p. 15]. The proofs of Theorems 3 and 4 are based on a result of Beurling [3, p. 20]. These results are sharp in some respects.

The special case $\psi(r)=\sigma\log^2 r$ in Theorem 1 gives the conclusion

$$\text{lower log-dens } E\{m(r)/M(r) > \exp(-(1+\epsilon)\delta^{-1}(2-\delta)\pi^2\sigma)\} \geqq 1-\delta.$$

The following argument in this special case is typical of the ideas used in proving Theorem 1, although this particular method does not lead to the best constants.

THEOREM 5. *If*

$$\limsup_{r\to\infty} \log M(r)/\log^2 r \leqq \sigma,$$

then, when $0<\delta<1$ *and* $\epsilon>0$,

$$\text{lower log-dens } E\{m(r)/M(r) > \exp(-(1+\epsilon)\delta^{-1}2\pi^2\sigma)\} \geqq 1-\delta.$$

PROOF. Suppose $f(0)=1$. Let $n(r)$ be the number of zeros of $f(z)$ in $|z|\leqq r$. Then by a well-known argument, $n(r)<4(\sigma+\epsilon)\log r$, for all $r\geqq r_0(\epsilon)$. If $\{r_n\}$ is the sequence of the moduli of the zeros of $f(z)$ arranged in order of increasing magnitude, we have

$$\log M(r)/m(r) \leqq \sum_{n=1}^{\infty} \log \frac{1 + r/r_n}{|1 - r/r_n|} \equiv \phi(r), \text{ say.}$$

The operations involved in the following estimation are justified by standard results about Lebesgue integration.

$$\int_0^r \phi(t)t^{-1}dt = \sum_{n=1}^{\infty} \left\{ \int_0^r \log \frac{1 + t/r_n}{|1 - t/r_n|} t^{-1}dt \right\}$$

$$= \sum_{r_n \leqq r} \left\{ 2 \sum_{m=0}^{\infty} (2m+1)^{-1} \left\{ \int_0^{r_n} (t/r_n)^{2m+1} t^{-1} dt + \int_{r_n}^{r} (r_n/t)^{2m+1} t^{-1} dt \right\} \right\}$$

$$+ \sum_{r_n > r} \left\{ 2 \sum_{m=0}^{\infty} (2m+1)^{-1} \int_0^r (t/r_n)^{2m+1} t^{-1} dt \right\}$$

$$= 2 \sum_{m=0}^{\infty} (2m+1)^{-1} \left\{ \sum_{r_n \leqq r} r_n^{-2m-1} \int_0^{r_n} t^{2m} dt + \sum_{r_n \leqq r} r_n^{2m+1} \int_{r_n}^{r} t^{-2m-2} dt \right.$$

$$\left. + \sum_{r_n > r} r_n^{-2m-1} \int_0^r t^{2m} dt \right\}$$

$$= 2 \sum_{m=0}^{\infty} (2m+1)^{-1} \left\{ \sum_{r_n \leqq r} (2m+1)^{-1} \right.$$

$$+ \sum_{r_n \leqq r} (2m+1)^{-1} r_n^{2m+1} (r_n^{-2m-1} - r^{-2m-1})$$

$$\left. + \sum_{r_n > r} (2m+1)^{-1} (r/r_n)^{2m+1} \right\}$$

$$= 2 \sum_{m=0}^{\infty} (2m+1)^{-1} \left\{ (2m+1)^{-1} n(r) \right.$$

$$+ \int_0^r (2m+1)^{-1} t^{2m+1} (t^{-2m-1} - r^{-2m-1}) dn(t)$$

$$\left. + \int_r^{\infty} (2m+1)^{-1} (r/t)^{2m+1} dn(t) \right\}$$

$$= 2 \sum_{m=0}^{\infty} (2m+1)^{-1} \left\{ r^{-2m-1} \int_0^r n(t) t^{2m} dt + r^{2m+1} \int_r^{\infty} n(t) t^{-2m-2} dt \right\}$$

$$< 16(\sigma + \epsilon) \sum_{m=0}^{\infty} (2m+1)^{-2} \log r, \qquad \text{for } r \geqq r_1(\epsilon),$$

i.e.,

$$\int_0^r \phi(t)t^{-1}dt < 2\pi^2(\sigma + \epsilon) \log r, \qquad r \geqq r_1(\epsilon).$$

Thus if, in a certain set S,

$$\phi(t) \geqq \delta^{-1} 2\pi^2(\sigma + \epsilon),$$

then for $r \geqq r_1(\epsilon)$,

$$\int_{(1,r)\cap S} t^{-1}dt < \delta \log r$$

so that

$$\text{upper log-dens } S \leqq \delta.$$

Theorem 5 follows at once from this.

Some of the results extend to subharmonic functions but a difference between sufficiently slowly growing subharmonic and integral functions is noted.

References

1. R. P. Boas, Jr., *Entire functions*, New York, Academic Press 1954.

2. W. K. Hayman, *Slowly growing integral and subharmonic functions*, Comm. Math. Helv. vol. 34 (1960) pp. 75–84.

3. B. Kjellberg, *On certain integral and harmonic functions*, Dissertation, Uppsala, 1948.

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