RESEARCH PROBLEMS

1. J. L. Brenner: *Group theory.*

In each of the alternating groups A_n, $4<n<10$, there is an element a_n such that every element g of the group is similar to a commutator of a_n: $g=u^{-1}(a_nya_n^{-1}y^{-1})u$, where u, y depend on g, but a_n does not. Find those alternating groups which enjoy this property. Some infinite alternating groups do, some do not. Those groups which enjoy this property are among the least complicated simple groups. (Received February 26, 1960.)

2. Olga Taussky: *Discuss the location of the eigenvalues of the Jordan product of two hermitian matrices.*

The product of two $n\times n$ hermitian matrices A and B is not in general hermitian and its eigenvalues are, in general, not even real unless one of the matrices is positive definite. The Jordan product $AB+BA$ is, however, hermitian. In order to study its eigenvalues it is permissible to assume that one of the matrices, A, say, is in diagonal form with eigenvalues $\alpha_1, \cdots, \alpha_n$. The Jordan product then coincides with the Hadamard-Schur product of the matrix $(\alpha_i+\alpha_k)$ and B. It was pointed out recently by M. Marcus and N. A. Khan (Canad. Math. Soc. Bull. vol. 2 (1959) pp. 81–83) that the Hadamard-Schur product of two $n\times n$ matrices $X=(x_{ik})$ and $Y=(y_{ik})$, i.e. the matrix $(x_{ik}y_{ik})$, is a principal minor of the Kronecker product of X and Y. Hence for hermitian factors the eigenvalues of the Hadamard-Schur product lie in the interval spanned by the products $\xi_i\eta_k$ where the ξ_i are the eigenvalues of X and the η_k are the eigenvalues of Y, each taken in any order. Since the eigenvalues of $(\alpha_i+\alpha_k)$ are easily computed, bounds for the eigenvalues of $AB+BA$ can be obtained. These bounds are, however, much too large while the maxima and minima of $2\alpha_i\beta_k$, which are bounds for commuting A and B, are not bounds in the general case (β_k being the eigenvalues of B). (Received September 29, 1959.)

3. F. H. Brownell: *Tauberian theorem problem.*

Let $F(\lambda)$ be a real valued function of real $\lambda\geqq 0$ which is of bounded variation over every finite interval $[0, N]$, which is continuous at $\lambda=0$ with $F(0)=0$, and which has $\int_0^\infty e^{-t\lambda}|dF(\lambda)|<+\infty$ for real $t>0$. With $s=t+iv$, t and v real, define $g(s)$ by the Lebesgue-Stieltjes integral $g(s)=\int_0^\infty e^{-s\lambda}dF(\lambda)$, analytic in $t>0$. Let F satisfy the conditions that

$$g(t) = b + O(\exp(-c/t)) \tag{1}$$

as $t \to 0^+$ for some real constants $c > 0$ and b, and that

$$F(\lambda) + K\lambda^\nu \tag{2}$$

be strictly increasing over $\lambda \geqq 1$ for some real constants $K > 0$ and $\nu \geqq 1$.

Based on the work of G. Freud, Ganelius [Kungl. Fysiogr. Sallsk. i Lund Förh. vol. 24 no. 20 (1954)] and Korevaar [Indag. Math. vol. 16 (1954) pp. 36–45] have shown that (1) and (2) here imply that as $\lambda \to +\infty$

$$F(\lambda) = O(\lambda^{\nu-1/2}). \tag{3}$$

PROBLEM. *Is it true, as conjectured, that* (3) *can be strengthened to*

$$F(\lambda) = o(\lambda^{\nu-1/2}) \tag{4}$$

as $\lambda \to +\infty$ *if in addition to* (1) *and* (2) *it is also assumed that* $g(iv) = \lim_{t\to 0^+} g(t+iv)$ *exists finite for all* $v \neq 0$, *that the resulting* $g(s)$ *is continuous in* $t \geqq 0$ *and* $s \neq 0$, *and that over all such* s

$$|g(s)| \leqq M_1 |s|^{-\nu+\eta} + M_2 \tag{5}$$

for some finite constants M_1 *and* M_2 *and* $\eta > 0$?

The example $F(\lambda) = \int_0^\lambda \sin(x^{1/2})x^{\nu-1}dx$, [Korevaar, Indag. Math. vol. 16 (1954) p. 43, Example 3.5)], for integer ν satisfies (1), (2), and (3) but not (4), proving (3) to be best possible under (1) and (2) only. Since this example also violates (5), the conjecture is still open.

The affirmative proof of this conjecture for $\nu = 1$ would give important information, namely

$$N_\gamma(\lambda) = \frac{\mu_2(D)}{4\pi}\lambda - (-1)^\gamma \frac{l(B)}{4\pi}\lambda^{1/2} + o(\lambda^{1/2}), \tag{6}$$

about the number $N_\gamma(\lambda)$ of eigenvalues $\leqq \lambda$ of the membrane equation for a bounded region D of the plane with polygonal boundary B, $\mu_2(D)$ being the area of D and $l(B)$ the sum of the perimeters of the polygons composing B, and $\gamma = 0$ or 1 denoting Dirichlet or Neumann boundary conditions respectively. For letting $F(\lambda)$ be the remainder in (6), (2) with $\nu = 1$ is automatic and (1) is known [Brownell, J. Math. Mech. vol. 6 (1957) pp. 119–166 and Bull. Amer. Math. Soc. vol. 63 no. 560 (1957) p. 284]; also (5) appears obtainable from the same analysis. The formula (6) (and more) is well known for rectangles, and its truth for general polygonal regions D would partially confirm a conjecture of Polya [C. R. Acad. Sci. Paris vol. 242 (1956) pp. 708–709]. (Received March 21, 1960.)