

MEASURE ALGEBRAS ON ABELIAN GROUPS

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The recent developments in the general field of Fourier analysis which I wish to describe, illustrate the algebraic point of view which has established itself here as well as in most other parts of analysis, and which has been most fruitful in suggesting new and interesting problems. Typically, a question will pose itself in algebraic terms, will then be transformed into a problem which can be attacked by the methods of classical analysis, and the algebraic aspect will reappear in the solution.

To cite just one example now, one may ask for the set of all automorphisms of the group algebra of the circle group; this turns out to be equivalent to the problem of finding all permutations of the set of all integers which carry Fourier series into Fourier series; the solution of this analytical problem involves arithmetic progressions, i.e., cosets of the subgroups of the additive group of the integers, and the group theoretic nature of the problem becomes apparent (the result is described in 3.3(d) below).

This is a large subject, and I will restrict myself to those topics with which I have had close personal contact. This choice will, of course, result in the omission of much that is interesting. In particular, I will say nothing about what is perhaps the most intriguing of the open problems in Fourier analysis: the problem of spectral synthesis, or, equivalently, the problem of finding all closed ideals in $L^1(G)$.

I. Preliminaries. 1.1. Unless the contrary is explicitly stated, any group mentioned in the sequel will be abelian and locally compact, with addition as group operation. Associated with every such group G there is a measure m (not identically 0 or ∞), the so-called *Haar measure* of G , with the following properties [6; 7; 21; 31]: m is a non-negative completely additive set function defined for all Borel sets in G , which is *regular* (i.e., $m(E) = \sup m(K) = \inf m(V)$, where K ranges over all compact subsets of E and V ranges over all open supersets of E), and which is *translation invariant*:

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$$(1.1.1) \quad m(E+x) = m(E) \quad (x \in G).$$

These conditions characterize m , up to a positive multiplicative constant. When G is compact, it is customary to normalize m so that $m(G) = 1$; when G is discrete, any set consisting of a single point of G is assigned the measure 1. In any case, $m(K)$ is finite if K is compact, $m(V)$ is positive if V is open, and $m(G)$ is finite if and only if G is compact.

1.2. DEFINITION OF $M(G)$. For any of the groups G under consideration, $M(G)$ is the set of all complex-valued completely additive Borel measures on G which are regular in the following sense: any measure μ with the above properties has a decomposition $\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$, with $\mu_j \geq 0$; it is required that each μ_j be regular in the sense in which this term was defined in 1.1. The requirement that $\mu(E)$ be a complex number is meant to exclude the possibility that $\mu(E) = \infty$. Thus the Haar measure of G belongs to $M(G)$ if and only if G is compact.

A norm is introduced in $M(G)$ by defining

$$(1.2.1) \quad \|\mu\| = \sup \sum |\mu(E_i)| \quad (\mu \in M(G)),$$

the sup being taken over all finite collections of pairwise disjoint Borel sets E_i whose union is G ; it is easy to see, via the Hahn decomposition theorem [7, p. 121], that $\|\mu\| < \infty$ for every $\mu \in M(G)$.

We say that μ is *concentrated on E* if $\mu(A) = 0$ whenever A and E are disjoint. If μ is concentrated on a countable set, then μ is *discrete*; if $\mu(E) = 0$ for every countable set E , then μ is *continuous*; if $\mu(E) = 0$ whenever $m(E) = 0$, where m is the Haar measure of G , then μ is *absolutely continuous*; finally, if μ is concentrated on a set E with $m(E) = 0$, then μ is *singular*.

1.3. DEFINITION OF $L^1(G)$. The set of all complex-valued functions on G which are integrable with respect to the Haar measure of G is denoted by $L^1(G)$. Every $f \in L^1(G)$ has associated with it an absolutely continuous measure $\mu_f \in M(G)$:

$$(1.3.1) \quad \mu_f(E) = \int_E f dm.$$

Conversely, the Radon-Nikodym theorem [7, p. 128] shows that every absolutely continuous $\mu \in M(G)$ is μ_f for some $f \in L^1(G)$. If, as is customary, we identify any two functions in $L^1(G)$ whose values differ only on a set of Haar measure zero, then the correspondence between f and μ_f is one-to-one, and we may therefore regard $L^1(G)$ as a subset of $M(G)$. It is convenient to use f and μ_f interchangeably.

The restriction of the norm defined by (1.2.1) to $L^1(G)$ coincides with the usual norm in spaces of integrable functions:

$$(1.3.2) \quad \|f\| = \|\mu_f\| = \int_G |f| \, dm \quad (f \in L^1(G)).$$

Note that $L^1(G) = M(G)$ if and only if G is discrete.

1.4. ALGEBRAIC OPERATIONS IN $M(G)$. It is obvious how $M(G)$ can be made into a linear space: if $\mu, \lambda \in M(G)$, define

$$(1.4.1) \quad \begin{aligned} (\mu + \lambda)(E) &= \mu(E) + \lambda(E), \\ (c\mu)(E) &= c \cdot \mu(E), \end{aligned}$$

for every Borel set E in G and for every complex number c . With these operations, and with the norm (1.2.1), $M(G)$ is in fact a Banach space, and $L^1(G)$ is a closed subspace of $M(G)$.

Multiplication of two measures is defined by convolution: If $\mu, \lambda \in M(G)$, let $\mu \times \lambda$ denote their product measure [7] in the Cartesian product space G^2 , and associate with each Borel set E in G the set

$$(1.4.2) \quad E_{(2)} = \{(x, y) \in G^2 \mid x + y \in E\}.$$

Put

$$(1.4.3) \quad (\mu * \lambda)(E) = (\mu \times \lambda)(E_{(2)}),$$

and call the set function $\mu * \lambda$ so defined the *convolution* of μ and λ .

It is not hard to verify that $\mu * \lambda \in M(G)$, that $\mu * \lambda = \lambda * \mu$ (since G is abelian), that convolution is associative, and that

$$(1.4.4) \quad \|\mu * \lambda\| \leq \|\mu\| \cdot \|\lambda\|;$$

each of these verifications is an application of Fubini's theorem.

Thus, with convolution as multiplication, $M(G)$ is a *Banach algebra*. There is a *unit element* in $M(G)$, namely the measure u such that $u(E) = 1$ if $0 \in E$ and $u(E) = 0$ otherwise; u is the unit mass concentrated at 0, and $\mu * u = \mu$ for every $\mu \in M(G)$.

The convolution of more than two measures can be defined by associativity; another (equivalent) definition, well adapted to some applications (see 2.3) is as follows: if $\mu_1, \dots, \mu_n \in M(G)$, associate with each $E \subset G$ the set

$$(1.4.5) \quad E_{(n)} = \{(x_1, \dots, x_n) \in G^n \mid x_1 + \dots + x_n \in E\},$$

and put

$$(1.4.6) \quad (\mu_1 * \mu_2 * \dots * \mu_n)(E) = (\mu_1 \times \mu_2 \times \dots \times \mu_n)(E_{(n)}),$$

where the measure on the right is the ordinary product measure.

We also define

$$(1.4.7) \quad \mu^1 = \mu, \quad \mu^n = \mu * \mu^{n-1}, \quad \mu^0 = u.$$

1.5. GROUP ALGEBRAS. By the Fubini theorem, (1.4.3) can be rewritten, for every Borel set E , in the form

$$(1.5.1) \quad (\mu * \lambda)(E) = \int_G \mu(E - x) d\lambda(x).$$

From this one immediately concludes the following: if μ is absolutely continuous, so is $\mu * \lambda$; if μ is continuous, so is $\mu * \lambda$; if both μ and λ are discrete, so is $\mu * \lambda$. In other words, $L^1(G)$ is a closed ideal in $M(G)$, and so is the set of all continuous $\mu \in M(G)$; the discrete measures form a closed subalgebra of $M(G)$.

The algebra $L^1(G)$ is usually called the *group algebra* of G , although this term should perhaps have been reserved for $M(G)$. In the classical cases (G finite) $L^1(G)$ and $M(G)$ coincide, and it is a matter of taste whether one wishes to consider the elements of a finite-dimensional group algebra as functions on G or as measures on G . However, $M(G)$ does have one advantage: its definition does not depend on the construction of Haar measure.

1.6. In this connection it may be of interest to note that the absolutely continuous measures in $M(G)$ can be characterized without any reference to Haar measure:

THEOREM. *A measure μ in $M(G)$ is absolutely continuous if and only if $\mu(E - x)$ is a continuous function of x , for every Borel set E in G .*

PROOF. Suppose $\mu = \mu_f$ for some $f \in L^1(G)$, and define f_x , for $x \in G$, by $f_x(y) = f(y - x)$. Then $\|f_x - f\| \rightarrow 0$ as $x \rightarrow 0$ [21, p. 118], and the continuity of $\mu(E - x)$ follows from the relation $\mu_f(E - x) = \mu_{f_x}(E)$.

To prove the converse, suppose $m(E) = 0$, and let g be the characteristic function of a Borel set A such that $m(A) < \infty$. Then

$$(1.6.1) \quad \int_A \mu(E - x) dm(x) = (g * \mu)(E).$$

Since $g \in L^1(G)$ and since $L^1(G)$ is an ideal in $M(G)$, the right member of (1.6.1) is 0. Hence $\mu(E - x) = 0$ for almost all x ; by continuity, $\mu(E) = 0$, and the theorem is proved.

1.7. CHARACTERS, DUAL GROUPS, FOURIER TRANSFORMS, AND FOURIER-STIELTJES TRANSFORMS. A *continuous character* on a group G is a continuous complex function γ on G , whose values will be denoted by (x, γ) , such that $|(x, \gamma)| = 1$ and

$$(1.7.1) \quad (x + y, \gamma) = (x, \gamma) \cdot (y, \gamma) \quad (x, y \in G).$$

The set of all such γ forms a group Γ , the *dual group* of G , if addition is defined by

$$(1.7.2) \quad (x, \gamma_1 + \gamma_2) = (x, \gamma_1) \cdot (x, \gamma_2) \quad (x \in G; \gamma_1, \gamma_2 \in \Gamma);$$

Γ is usually given the compact-open topology: to every compact set K in G and to every $\epsilon > 0$ there corresponds a neighborhood of 0 which consists of all γ such that

$$(1.7.3) \quad |1 - (x, \gamma)| < \epsilon \quad (x \in K).$$

With this topology, Γ is a locally compact abelian group, and G is in turn the dual group of Γ (this is the celebrated Pontryagin duality theorem [3]). If G is compact then Γ is discrete, and vice versa.

With every $f \in L^1(G)$ is associated its *Fourier transform* \hat{f} :

$$(1.7.4) \quad \hat{f}(\gamma) = \int_G (-x, \gamma) f(x) dm(x) \quad (\gamma \in \Gamma);$$

if $\mu \in M(G)$, its *Fourier-Stieltjes transform* (abbreviated to F. S. transform in the sequel) is the function $\hat{\mu}$ defined analogously by

$$(1.7.5) \quad \hat{\mu}(\gamma) = \int_G (-x, \gamma) d\mu(x) \quad (\gamma \in \Gamma).$$

It is clear that (1.7.4) and (1.7.5) coincide for absolutely continuous μ : in the notation of 1.3, $\hat{f} = \hat{\mu}_f$.

A concise presentation of the basic theorems about these transforms may be found in [3]. The functions $\hat{\mu}$ are uniformly continuous and bounded on Γ , the correspondence between μ and $\hat{\mu}$ is one-to-one, and the mapping $\mu \rightarrow \hat{\mu}$ is thus an isomorphism of the algebra $M(G)$ onto an algebra of continuous functions on Γ : the mapping is clearly linear, and if $\sigma = \mu * \lambda$, then

$$(1.7.6) \quad \hat{\sigma}(\gamma) = \hat{\mu}(\gamma) \hat{\lambda}(\gamma) \quad (\gamma \in \Gamma).$$

It follows that the mapping $\mu \rightarrow \hat{\mu}(\gamma)$, for any fixed $\gamma \in \Gamma$, is a complex homomorphism of $M(G)$, and the fact that $\hat{\mu}$ determines μ implies that $M(G)$ is a semi-simple Banach algebra [21, p. 76].

1.8. THE GELFAND TRANSFORMS OF MEASURES. Being a commutative Banach algebra with unit, $M(G)$ has a compact maximal ideal space Δ . We may consider Δ to be the set of all homomorphisms h of $M(G)$ onto the complex field; the topology of Δ is the weakest one in which the functions $\hat{\mu}$ defined on Δ by

$$(1.8.1) \quad \hat{\mu}(h) = h(\mu) \quad (\mu \in M(G), h \in \Delta)$$

are continuous. We call $\hat{\mu}$ the *Gelfand transform* of μ .

We saw in 1.7 that every $\gamma \in \Gamma$ defines an element h_γ of Δ , by the formula

$$(1.8.2) \quad h_\gamma(\mu) = \hat{\mu}(\gamma) = \int_G (-x, \gamma) d\mu(x) \quad (\mu \in M(G)).$$

We identify γ and h_γ , and thus embed Γ in Δ ; this embedding is a homeomorphism of Γ into Δ , and the Fourier-Stieltjes transforms on Γ are precisely the restrictions of the Gelfand transforms to Γ (so that the two ways in which the notation $\hat{\mu}$ was introduced are compatible).

1.9. THE CIRCLE GROUP AND THE INTEGERS. To cite just one familiar example, let G be the circle group C , i.e., the set of all complex numbers of absolute value 1, with multiplication as group operation (or, equivalently, the set of all real numbers, with addition mod 2π). The dual group of C is J , the additive group of all integers; the characters are of the form

$$(1.9.1) \quad (e^{i\theta}, n) = e^{in\theta} \quad (e^{i\theta} \in C, n \in J),$$

and the F. S. transform $\hat{\mu}$ of any $\mu \in M(C)$ is given by

$$(1.9.2) \quad \hat{\mu}(n) = \int_0^{2\pi} e^{-in\theta} d\mu(\theta) \quad (n \in J);$$

the numbers $\hat{\mu}(n)$ are usually called the F. S. coefficients of μ . The Haar measure of C is the ordinary Lebesgue measure, divided by 2π .

If $G = J$, then $\Gamma = C$, $(n, e^{i\theta}) = e^{in\theta}$, $M(J) = L^1(J)$, and if $\mu \in M(J)$, then

$$(1.9.3) \quad \hat{\mu}(e^{i\theta}) = \sum_{n=-\infty}^{\infty} \mu(\{n\}) e^{-in\theta}.$$

In other words, the F. S. transforms on C are precisely those functions which are sums of absolutely convergent trigonometric series.

II. The maximal ideal space of $M(G)$. 2.1. We saw in 1.8 that Γ is a subset of Δ , the maximal ideal space of $M(G)$. In one important case, $\Gamma = \Delta$: this happens when G is discrete, for then $M(G) = L^1(G)$, and it is known [21, p. 136] that Γ is always the maximal ideal space of $L^1(G)$. If G is not discrete, however, then Γ is not compact, and hence Γ is a proper subset of Δ . The question arises whether one can say anything more specific about the relation between Γ and Δ .

2.2. In 1950, Šreider [30] discovered a surprising fact: if G is the additive group of the real line, then Γ is not dense in Δ , and $M(G)$ is not symmetric (i.e., the set of all Gelfand transforms is not closed

under complex conjugation). This was all the more surprising since the F. S. transforms are *always* closed under conjugation: if $\sigma(E) = \bar{\mu}(-E)$, then $\hat{\sigma}(\gamma)$ is the complex conjugate of $\hat{\mu}(\gamma)$, for all $\gamma \in \Gamma$.

Hewitt [13] extended this result to groups G in which every neighborhood of 0 contains an element of infinite order; the essential part of his proof was simplified in [25]. Recently Williamson [37] has succeeded in proving the same thing for every nondiscrete G . I will sketch a proof of this interesting result, using the method of [25] to prove the lemma:

2.3. LEMMA. *On every nondiscrete locally compact abelian group G there is a non-negative continuous measure $\mu \in M(G)$, such that $\mu(E) = \mu(-E)$, $\|\mu\| = 1$, and*

$$(2.3.1) \quad \left\| \sum_{i=0}^n a_i \mu^i \right\| = \sum_{i=0}^n |a_i|$$

for every choice of complex numbers a_0, \dots, a_n .

(The measures μ^i are defined by (1.4.7).)

PROOF. One of two cases must occur [25]: either every neighborhood of 0 in G contains an element of infinite order, or G contains a compact open subgroup of bounded order. In the former case, put $W(G) = \infty$. In the latter case, there is a compact open group K in G and there is an integer $W(G) \geq 2$, such that (i) the order of no element of K exceeds $W(G)$, (ii) every neighborhood of 0 in G contains an element of order $W(G)$. With this notation, the following proposition holds:

There exists a set Q in G , homeomorphic to the Cantor set, with the following property: if g_1, \dots, g_p are distinct points of Q , if n_1, \dots, n_p are integers, not all 0, such that $|n_i| < W(G)$, then

$$(2.3.2) \quad n_1 g_1 + \dots + n_p g_p \neq 0.$$

For $W(G) = \infty$, this was proved in [25]. In the other cases the proof is quite analogous; Q is constructed as a subset of K , by a suitable imitation of the usual construction of a Cantor set on the line as the intersection of sets E_n which are unions of 2^n intervals.

Having Q , we put $P = Q \cup (-Q)$, and take for μ any continuous non-negative measure which is concentrated on P , such that $\|\mu\| = 1$ and $\mu(E) = \mu(-E)$; since $P = -P$, this can be done. Since μ^n is concentrated on P_n , where $P_1 = P$ and $P_{n+1} = P_n + P$, it is enough to show that $m < n$ implies

$$(2.3.3) \quad \mu^n(P_m) = 0.$$

Fix $m < n$, and let S be the set of all $(x_1, \dots, x_n) \in G^n$ such that

$$(2.3.4) \quad x_1 \in P, \dots, x_n \in P; \quad x_1 + \dots + x_n \in P_m.$$

Define $\mu_{(n)}$ by: $\mu_{(1)} = \mu$, $\mu_{(n)} = \mu \times \mu_{(n-1)}$, so that $\mu_{(n)}$ is a measure on G^n . Since μ is concentrated on P , (1.4.6) shows that

$$(2.3.5) \quad \mu^n(P_m) = \mu_{(n)}(S).$$

Suppose $(x_1, \dots, x_n) \in S$. By (2.3.4) there are points $y_1, \dots, y_m \in P$, such that

$$(2.3.6) \quad x_1 + \dots + x_n = y_1 + \dots + y_m.$$

The definition of P shows that $x_i = \epsilon_i g_i$, with $\epsilon_i = \pm 1$, $g_i \in Q$. If $g_i \neq g_j$ whenever $i \neq j$, the fact that $m < n$ leads to a relation which contradicts (2.3.2). Hence $x_i \pm x_j = 0$ for some $i \neq j$. Since μ is continuous, the set of all points in $G^{(n)}$ at which any of the relations $x_i \pm x_j = 0$ holds, has measure 0 with respect to $\mu_{(n)}$. It follows that $\mu_{(n)}(S) = 0$, and the lemma is proved.

2.4. THEOREM [37]. *Suppose G is a locally compact, abelian, non-discrete group. Then*

- (a) $M(G)$ is not symmetric;
- (b) the closure of Γ in Δ does not contain the Šilov boundary of Δ ;
- (c) the Wiener-Pitt phenomenon occurs.

For the definition of Šilov boundary we refer to [21, p. 80]. By (c) we mean that there exists a F. S. transform on Γ whose reciprocal is bounded but is not a F. S. transform; on the real line such an example was constructed in [36]. We note that each of the three assertions of the theorem implies that Γ is not dense in Δ .

PROOF. Take μ as in Lemma 2.3, and put $\sigma = u - \mu^2$, where u is the unit of $M(G)$. By (2.3.1),

$$\|\sigma^n\| = \left\| \sum_0^n \binom{n}{k} (-1)^k \mu^{2k} \right\| = \sum_0^n \binom{n}{k} = 2^n \quad (n = 1, 2, 3, \dots),$$

so that the spectral norm of σ , $\lim \|\sigma^n\|^{1/n}$, is 2 [21, p. 75]. Thus there is an $h \in \Delta$ such that $|h(\sigma)| = 2$. Since $\|\mu\| = 1$, we have $|h(\mu^2)| \leq 1$, and the equation $|1 - h(\mu^2)| = 2$ implies that $h(\mu^2) = -1$ and $h(\sigma) = 2$.

Since $\mu(-E) = \mu(E)$ and μ is real, $\hat{\mu}(\gamma)$ is real on Γ . However, $\hat{\mu}(h) = h(\mu) = \pm i$. If the complex conjugate of $\hat{\mu}$ were the Gelfand transform of some $\nu \in M(G)$, then $\hat{\mu} - \hat{\nu}$ would vanish at every point of Γ , so that $\mu = \nu$. But $\hat{\mu}(h) \neq \hat{\nu}(h)$. This contradiction proves (a).

Since $0 \leq \hat{\mu}^2(\gamma) \leq 1$ on Γ , and $\hat{\sigma} = 1 - \hat{\mu}^2$, we have $0 \leq \hat{\sigma}(\gamma) \leq 1$ on Γ ; since $\hat{\sigma}(h) = 2$, $|\hat{\sigma}|$ does not attain its maximum over Δ at any point of the closure of Γ , and (b) follows.

To prove (c), put $\tau = u + \mu^2$. Then $1 \leq \hat{\tau}(\gamma) \leq 2$ on Γ , but $\tau(h) = 1 + h(\mu^2) = 0$, so that τ^{-1} does not exist in $M(G)$.

This completes Theorem 2.4.

2.5. There is not much known about the topological properties of Δ (Šreider's description of the points of Δ by means of "generalized characters" [30] seems to throw little light on the problem), but here are a few observations (it is assumed throughout that G is not discrete).

(a) *The Gelfand transforms of the absolutely continuous measures in $M(G)$ vanish outside Γ .*

If h does not annihilate every $f \in L^1(G)$, then there is a $\gamma \in \Gamma$ such that $h(f) = \hat{f}(\gamma)$ for all $f \in L^1(G)$, and $\hat{f}_0(\gamma) = 1$ for some $f_0 \in L^1(G)$. For any $\mu \in M(G)$, the measure $\sigma = \mu * f_0$ is absolutely continuous, and so $h(\mu) = h(\mu)h(f_0) = h(\sigma) = \hat{\sigma}(\gamma) = \hat{\mu}(\gamma)$. In other words, $h \in \Gamma$.

(b) COROLLARY. Γ is an open subset of Δ .

(c) Although Γ is not dense in Δ , Δ is "closely attached" to Γ in the following sense:

Δ is not the union of two nonempty open-closed sets, one of which contains Γ .

If it were, a well-known theorem of Šilov [29] would imply the existence of some $\mu \in M(G)$ such that $\hat{\mu}(\gamma) = 0$ on Γ , but $\hat{\mu}(h) = 1$ on some nonempty subset of $\Delta - \Gamma$, which is impossible.

(d) COROLLARY. If Γ is connected, so is Δ .

(e) *There are singular measures in $M(C)$ whose Gelfand transforms vanish outside J .* I.e., the converse of (a) is not true. (See 1.9 for notation.)

Indeed, Salem [28, Theorem III] has shown that there are singular measures $\mu \in M(C)$ whose F. S. coefficients are $O(|n|^{-\alpha})$ for some $\alpha > 0$; hence $\mu^k \in L^2$ for some positive integer k , and (a) implies that $[h(\mu)]^k = h(\mu^k) = 0$ outside J .

(f) *There is a singular measure $\mu \in M(C)$ such that $\hat{\mu}(n) \rightarrow 0$ as $|n| \rightarrow \infty$, but such that $\hat{\mu}(h) \neq 0$ for some $h \in \Delta - J$.* (Here Δ is of course the maximal ideal space of $M(C)$.)

It is known that there is a perfect set P on C which is not a basis and which carries a positive continuous measure μ whose F. S. coefficients tend to 0 as $|n| \rightarrow \infty$ [30, p. 25; 28, Theorem IV]. Let F be the regular family of sets (in the sense of Raikov [30, p. 15; 5, pp. 184-187]) generated by P , let σ_F be the projection of σ on F , and define $h(\sigma) = \sigma_F(C)$. Then $h \notin J$, but $h(\mu) > 0$.

2.6. The following problems might be of interest:

(a) What is the Šilov boundary of Δ ? Is it all of Δ ?

(b) Is there any subset S of Δ , larger than the closure of Γ , on which the restrictions of the Gelfand transforms are closed under conjugation?

(c) Let $R(G)$ be the algebra of all self-adjoint elements of $M(G)$. That is, $\mu \in R(G)$ if and only if there is a $\sigma \in M(G)$ whose Gelfand transform is the complex conjugate of the Gelfand transform of μ . $R(G)$ contains all μ such that $\hat{\mu} = 0$ outside Γ (or outside any set S with the properties (b)); $R(G)$ also contains all discrete measures on G . Is there any intrinsic description of those measures which belong to $R(G)$?

III. Isomorphisms of group algebras. 3.1. Consider two locally compact abelian groups, G_1 and G_2 , with dual groups Γ_1 and Γ_2 , and suppose that τ is an isomorphism of G_2 onto G_1 (an isomorphism of two topological groups is of course also required to be a homeomorphism). Associated with τ there is an isomorphism ϕ of Γ_2 onto Γ_1 , defined by

$$(3.1.1) \quad (x, \phi(\gamma)) = (\tau^{-1}(x), \gamma) \quad (x \in G_1, \gamma \in \Gamma_2),$$

and there is a positive constant k such that

$$(3.1.2) \quad m_1(\tau E) = k \cdot m_2(E),$$

where m_i denotes the Haar measure of G_i (this depends on the uniqueness theorem for Haar measure).

Fix some $\gamma_0 \in \Gamma_2$, and let T be the mapping of $M(G_1)$ into $M(G_2)$ defined by

$$(3.1.3) \quad (T\mu)(E) = \int_{\tau E} (\tau^{-1}(y), \gamma_0) d\mu(y) \quad (\mu \in M(G_1)).$$

If we substitute an absolutely continuous μ into (3.1.3), so that $d\mu(x) = f(x) dm_1(x)$ for some $f \in L^1(G_1)$, a simple computation shows that T carries $L^1(G_1)$ into $L^1(G_2)$, and that

$$(3.1.4) \quad (Tf)(x) = k \cdot (x, \gamma_0) \cdot f(\tau(x)) \quad (x \in G_2, f \in L^1(G_1)).$$

The Fourier transforms of (3.1.3) and (3.1.4) are easily seen to be

$$(3.1.5) \quad \begin{cases} (T\mu)^\wedge(\gamma) = \hat{\mu}(\phi(\gamma) - \phi(\gamma_0)), \\ (Tf)^\wedge(\gamma) = f(\phi(\gamma) - \phi(\gamma_0)) \end{cases} \quad (\mu \in M(G_1), f \in L^1(G_1), \gamma \in \Gamma_2).$$

Since ϕ is an isomorphism onto Γ_1 , (3.1.5) shows that T is an isomorphism of $M(G_1)$ onto $M(G_2)$ whose restriction to $L^1(G_1)$ is an isomorphism onto $L^1(G_2)$; this follows from the obvious linearity of T , combined with the identity

$$(3.1.6) \quad (T\mu * T\lambda) = T(\mu * \lambda) \quad (\mu, \lambda \in M(G_1)),$$

which is an immediate consequence of (3.1.5). Moreover, T is an isometry:

$$(3.1.7) \quad \|T\mu\| = \|\mu\| \quad (\mu \in M(G_1)).$$

This is a consequence of (3.1.3).

Conversely, if ϕ is any isomorphism of Γ_2 onto Γ_1 , then (3.1.5) induces an isomorphism T between the algebras $M(G_i)$ and also between the algebras $L^1(G_i)$; ϕ determines τ via (3.1.1), and T satisfies (3.1.3), (3.1.4), and (3.1.7).

The following questions now arise quite naturally: Are there any other isomorphisms between these group algebras? Can nonisomorphic groups have isomorphic group algebras? If the answer is affirmative (a simple example of this sort, in which the groups are finite, was given by Wendel [32]), to what extent does the structure of $L^1(G)$ or of $M(G)$ determine the structure of G ?

3.2. Suppose, then, that T is an isomorphism of $L^1(G_1)$ onto $L^1(G_2)$. Since Γ_i is the maximal ideal space of $L^1(G_i)$, T induces a homeomorphism α of Γ_2 onto Γ_1 , such that

$$(3.2.1) \quad (Tf)^\wedge(\gamma) = \hat{f}(\alpha(\gamma)) \quad (f \in L^1(G_1), \gamma \in \Gamma_2).$$

Helson has shown that T can be extended to an isomorphism of $M(G_1)$ onto $M(G_2)$ [8]; his proof depends on the following characterization of the F. S. transforms: *a function g on Γ_i is a F. S. transform if and only if the mapping $\hat{f} \rightarrow g \cdot \hat{f}$ carries every Fourier transform to a Fourier transform.* The action of T on $M(G_1)$ is described by

$$(3.2.2) \quad (T\mu)^\wedge(\gamma) = \hat{\mu}(\alpha(\gamma)) \quad (\mu \in M(G_1), \gamma \in \Gamma_2).$$

Put $\mu_x = Te_x$, where, for every $x \in G_1$, e_x is the measure of mass 1 concentrated at the point x . Since $e_x * e_y = e_{x+y}$, we also have

$$(3.2.3) \quad \mu_{x+y} = \mu_x * \mu_y \quad (x, y \in G_1),$$

and since T is an isomorphism between two semi-simple Banach algebras, T is bounded [21, p. 76]. We conclude that T induces a bounded homomorphism $x \rightarrow \mu_x$ of G_1 into $M(G_2)$.

In terms of α , the conclusion is (by (3.2.2))

$$(3.2.4) \quad \hat{\mu}_x(\gamma) = (-x, \alpha(\gamma)) \quad (x \in G_1, \gamma \in \Gamma_2).$$

3.3. The following information has so far been obtained by determining all bounded homomorphisms $x \rightarrow \mu_x$ or, equivalently, by investigating for which homeomorphisms α the functions $(-x, \alpha(\gamma))$ are F. S. transforms of uniformly bounded measures μ_x :

(a) If Γ_1 (hence also Γ_2) is connected, then α is an isomorphism of Γ_2 onto Γ_1 , followed by a translation of Γ_1 [1].

The simple isomorphisms discussed in 3.1 are thus the only ones in this situation, and we can state a corollary:

If $L(G_1)$ and $L(G_2)$ are isomorphic, and if Γ_1 is connected, then G_1 and G_2 are isomorphic.

The crucial step in the proof of (a) is the analysis of (3.2.4) if Γ_2 is the real line [1]:

(b) Let f be a function on the real line, and suppose there are measures μ_n on the line, with $\{\|\mu_n\|\}$ bounded and $\hat{\mu}_n = f^n$, for $n = 0, \pm 1, \pm 2, \dots$. Then there exist real numbers a and b such that

$$(3.3.1) \quad f(t) = e^{i(at+b)} \quad (-\infty < t < \infty).$$

Note that the mapping $t \rightarrow at + b$ (with $a \neq 0$) is an automorphism of the line, followed by a translation.

(c) If $\|Tf\| \leq \|f\|$ for all $f \in L^1(G_1)$, then T is of the form (3.1.4).

This is proved in [8]; Wendel [33] has obtained this result even for noncommutative locally compact groups.

COROLLARY. If there is an isometric isomorphism between $L^1(G_1)$ and $L^1(G_2)$, then G_1 and G_2 are isomorphic.

(d) An analysis of (3.2.3) in the case $G_1 = G_2 = C$ (the circle group) has led to the following result [23] (J again denotes the integer group):

A one-to-one mapping α of J onto J induces an automorphism of $L^1(C)$ (and of $M(C)$) if and only if there is a mapping β of J onto J which differs from α in only a finite number of places, such that

$$(3.3.2) \quad \beta(n+g) + \beta(n-g) = 2\beta(n)$$

for some $g > 0$ and all $n \in J$.

In other words, the series $\sum_{-\infty}^{\infty} c(\alpha(n))e^{in\theta}$ is a Fourier series whenever $\sum_{-\infty}^{\infty} c(n)e^{in\theta}$ is a Fourier series, if and only if α is of the above form (and similarly for F. S. series). The endomorphisms of $L^1(C)$ are also determined in [23].

The mappings β which satisfy (3.3.2) can be described in a different way: suppose

$$(3.3.3) \quad J = A_1 \cup \dots \cup A_k = B_1 \cup \dots \cup B_k,$$

where each A_i and each B_i is an infinite arithmetic progression, and $A_i \cap A_j = B_i \cap B_j = 0$ if $i \neq j$. Let β map A_i linearly onto B_i in any way whatsoever (the action of β on A_i is independent of its action on A_j). Then β satisfies (3.3.2), and every β satisfying (3.3.2) is obtained in this way.

(e) A slight modification of the work in [23] establishes the following [26]:

$L^1(G)$ and $L^1(C)$ are isomorphic if and only if G is a direct sum $C + F$, where F is a finite abelian group.

If $G = C + F$, the isomorphisms of $L^1(G)$ onto $L^1(C)$ are again induced by mappings α of J onto Γ which are of the form described in (d).

3.4. Let T now be an isomorphism of $M(G_1)$ onto $M(G_2)$; since Δ_i is the maximal ideal space of $M(G_i)$, T induces a homeomorphism α of Δ_2 onto Δ_1 , such that

$$(3.4.1) \quad (T\mu) \wedge (\gamma) = \hat{\mu}(\alpha(\gamma)) \quad (\mu \in M(G_1), \gamma \in \Delta_2).$$

We saw in 3.2 that every isomorphism of $L^1(G_1)$ onto $L^1(G_2)$ extends to $M(G_1)$. Is it true, conversely, that the restriction of T to $L^1(G_1)$ maps $L^1(G_1)$ onto $L^1(G_2)$? (Or, is it true that every automorphism of $M(G)$ preserves $L^1(G)$?) For this it is clearly necessary that the α defined by (3.4.1) should map Γ_2 onto Γ_1 . *But this necessary condition is also sufficient:*

Let $M_c(G_i)$ be the set of all $\mu \in M(G_i)$ such that $\hat{\mu}$ vanishes outside some compact subset of Γ_i . If $\mu \in M_c(G_i)$, then $\hat{\mu} \in L^2(\Gamma_i)$. It follows that $M_c(G_i) \subset L^1(G_i)$; moreover, $M_c(G_i)$ is dense in $L^1(G_i)$. Thus T maps $M_c(G_1)$ onto $M_c(G_2)$ if $\alpha(\Gamma_2) = \Gamma_1$, and this implies that T maps $L^1(G_1)$ onto $L^1(G_2)$.

We next observe that if α is given by (3.4.1), then $\alpha(\overline{\Gamma_2}) = \overline{\Gamma_1}$. ($\overline{\Gamma}$ denotes the closure of Γ in Δ .) This follows from the fact that $\overline{\Gamma_i}$ is the unique closed subset E of Δ_i which is minimal with respect to the following property: if $\sigma \in M(G_i)$ and if $\hat{\sigma} = 0$ at every point of E , then $\sigma = 0$.

If we now assume that Γ_1 satisfies the first axiom of countability, we can prove that $\alpha(\Gamma_2) = \Gamma_1$ (this is undoubtedly true without any countability restriction, but I have no proof that covers the general case): Since $\alpha(\overline{\Gamma_2}) = \overline{\Gamma_1}$, there is a dense open subset V of Γ_2 such that $\alpha(V) \subset \Gamma_1$, and for any $\gamma_0 \in \Gamma_2$ there is a simple countable sequence $\{\gamma_n\}$, with $\gamma_n \in V$, such that $\gamma_n \rightarrow \gamma_0$. By (3.4.1),

$$(3.4.2) \quad \hat{\mu}(\alpha(\gamma_n)) \rightarrow \hat{\mu}(\alpha(\gamma_0)) \quad (\mu \in M(G_1)).$$

In particular, if $\mu = e_x$ (see 3.2), then

$$(3.4.3) \quad (-x, \alpha(\gamma_n)) \rightarrow (-x, \alpha(\gamma_0)) \quad (x \in G_1).$$

If $\{\alpha(\gamma_n)\}$ lies in a compact subset of Γ_1 , then $\alpha(\gamma_0) \in \Gamma_1$. If not, there is a subsequence, again denoted by $\{\alpha(\gamma_n)\}$, which tends to infinity in Γ_1 . Integrating (3.4.3) over some set A of positive finite

Haar measure, and applying the Riemann-Lebesgue theorem, we obtain

$$(3.4.4) \quad \int_A (-x, \alpha(\gamma_n)) dm_1(x) \rightarrow 0 \quad (n \rightarrow \infty);$$

by the Lebesgue convergence theorem (and this is where a simple countable sequence is needed), (3.4.3) implies

$$(3.4.5) \quad \int_A (-x, \alpha(\gamma_0)) dm_1(x) = 0.$$

Since A was arbitrary, $(-x, \alpha(\gamma_0)) = 0$ for almost all x , contradicting (3.4.3).

Thus $\alpha(\gamma_0) \in \Gamma_1$. It follows that $\alpha(\Gamma_2) \subset \Gamma_1$. Since the same argument applies to α^{-1} , the following theorem is proved:

3.5. THEOREM. *If T is an isomorphism of $M(G_1)$ onto $M(G_2)$ and if Γ_1 satisfies the first axiom of countability, then T maps $L^1(G_1)$ onto $L^1(G_2)$.*

The structure theorem for locally compact abelian groups [31, p. 110] shows that the countability assumption imposed on Γ_1 is equivalent to the following condition on G_1 : There is an open-closed subgroup G_0 of G_1 which is the direct sum of a Euclidean space and a compact group, and which has only a countable number of cosets in G_1 . In other words, the Haar measure of G_1 is σ -finite.

3.6. In 3.2, a condition was found (namely (3.2.4)) which is necessary for the homeomorphism α to induce an isomorphism of $L^1(G_1)$ onto $L^1(G_2)$, via (3.2.1). But this condition is also sufficient, as shown by the following theorem, in conjunction with §3.4:

THEOREM. *Let α be a mapping (not assumed to be continuous) of Γ_2 into Γ_1 , with the following property: for each $x \in G_1$ there is a $\mu_x \in M(G_2)$ such that*

$$(3.6.1) \quad \hat{\mu}_x(\gamma) = (-x, \alpha(\gamma)) \quad (x \in G_1, \gamma \in \Gamma_2),$$

and $\|\mu_x\| \leq A < \infty$, where A is independent of x . Then for every $\sigma \in M(G_1)$, the function f defined by

$$(3.6.2) \quad f(\gamma) = \hat{\sigma}(\alpha(\gamma)) = \int_{G_1} (-x, \alpha(\gamma)) d\sigma(x) \quad (\gamma \in \Gamma_2)$$

is a Fourier-Stieltjes transform on Γ_2 .

PROOF. Choose $\gamma_1, \dots, \gamma_p \in \Gamma_2$ and choose complex numbers a_1, \dots, a_p . Then

$$\begin{aligned}
\sum_n a_n f(\gamma_n) &= \sum_n a_n \hat{\sigma}(\alpha(\gamma_n)) \\
&= \int_{G_2} \sum_n a_n \hat{\mu}_x(\gamma_n) d\sigma(x) \\
&= \int_{G_2} \left\{ \int_{G_2} \sum_n a_n(-y, \gamma_n) d\mu_x(y) \right\} d\sigma(x),
\end{aligned}$$

so that

$$(3.6.3) \quad \left| \sum_n a_n f(\gamma_n) \right| \leq \|\sigma\| \cdot A \cdot \sup_n \left| \sum_n a_n(y, \gamma_n) \right| \quad (y \in G_2).$$

Since this holds for arbitrary $\gamma_1, \dots, \gamma_p$ and a_1, \dots, a_p , f is a F. S. transform, by a theorem of Eberlein [4], whose original form is due to Bochner [2].

IV. Idempotent measures. 4.1. The discussion of isomorphisms of group algebras has led us to the problem of determining the bounded homomorphisms of G_1 into $M(G_2)$, given by

$$(4.1.1) \quad \mu_{x+y} = \mu_x * \mu_y \quad (x, y \in G_1).$$

A special case of this problem deserves attention: if $x=y=0$, (4.1.1) becomes

$$(4.1.2) \quad \mu = \mu^2.$$

That is to say, we are led to the problem of finding all idempotents in $M(G)$; clearly, we cannot hope to solve (4.1.1) unless we can solve (4.1.2).

(a) This problem has two other aspects which make it interesting: There is a one-to-one correspondence between the set of all bounded projections P of $L^1(G)$ which commute with translations, and the set of all idempotent $\mu \in M(G)$, given by

$$(4.1.3) \quad Pf = f * \mu.$$

(b) One of the basic problems of Fourier analysis is that of determining just which functions on Γ are F. S. transforms. If we know all idempotents in $M(G)$, then we know all simple functions (i.e., those with only a finite set of values) which are F. S. transforms on Γ .

4.2. If $\mu \in M(G)$ and $\mu = \mu^2$, then $\hat{\mu} = \hat{\mu}^2$, so that $\hat{\mu}(\gamma) = 0$ or 1 for all $\gamma \in \Gamma$. Let $S(\mu)$ be the set of all $\gamma \in \Gamma$ at which $\hat{\mu}(\gamma) = 1$. The problem of finding all idempotents in $M(G)$ is thus equivalent to that of find-

ing all subsets of Γ whose characteristic functions are F. S. transforms.

The continuity of $\hat{\mu}$ implies that $S(\mu)$ must be open-closed. A particular case of the problem can therefore be solved immediately: if Γ is connected, then $M(G)$ has no nontrivial idempotents (there are always two trivial ones: $\mu=0$ and $\mu=u$, the unit of $M(G)$).

It is interesting that the problem reduces to the consideration of compact G :

4.3. THEOREM [27]. *If $\mu \in M(G)$ and if $\mu = \mu^2$, then μ is concentrated on a compact subgroup K of G (and hence μ may be regarded as an element of $M(K)$).*

4.4. We may thus suppose that G is compact. If Λ is a subgroup of Γ and H is the subgroup of G which consists of all x such that $(x, \gamma) = 1$ for all $\gamma \in \Lambda$, then m_H (the Haar measure of H) is an idempotent element of $M(G)$, and $\Lambda = S(m_H)$. If we define μ by

$$(4.4.1) \quad d\mu(x) = (x, \gamma) dm_H(x)$$

for some $\gamma \in \Gamma$, then $\mu = \mu^2$, and $S(\mu) = \Lambda + \gamma$. Thus every coset of Λ is $S(\mu)$ for some idempotent μ .

Let $R(\Gamma)$ be the smallest family of subsets of Γ which contains all cosets of all subgroups of Γ , and which is closed under complementation and finite intersection. We call $R(\Gamma)$ the *coset ring* of Γ .

Since $S(\mu) \cap S(\lambda) = S(\mu * \lambda)$ and $S(u - \mu)$ is the complement of $S(\mu)$, the above remark implies that every set in $R(\Gamma)$ is $S(\mu)$ for some idempotent $\mu \in M(G)$. The converse is probably true too:

4.5. CONJECTURE. *A set $E \subset \Gamma$ is $S(\mu)$ for some idempotent $\mu \in M(G)$ if and only if E belongs to $R(\Gamma)$.*

4.6. If $G = C$, this conjecture is true, as was proved by Helson [9; 10] (his result preceded and suggested the conjecture) and I have recently extended it to all finite-dimensional torus groups [27], using the decomposition into irreducible measures described in 4.7, as well as some work on multiple Fourier series which was done by Helson and Lowdenslager [12].

4.7. I call an idempotent $\mu \in M(G)$ *irreducible* if the smallest compact subgroup H of G on which μ is concentrated has the following property: if K is a compact subgroup of H and if $m(K) = 0$, where m is the Haar measure of H , then $\mu(E) = 0$ for all subsets E of K .

The above-mentioned decomposition theorem states: *if $\mu \in M(G)$ and $\mu = \mu^2$, then there are integers a_1, \dots, a_n and irreducible idempotents μ_1, \dots, μ_n in $M(G)$, such that $\mu = a_1\mu_1 + \dots + a_n\mu_n$.*

In order to prove the conjecture in its full generality, one has there-

fore to show that every irreducible idempotent measure μ is absolutely continuous with respect to the Haar measure of the smallest compact group on which μ is concentrated.

4.8. One other special case should be mentioned: If $\mu \in M(G)$, $\mu = \mu^2$, and $\|\mu\| = 1$, then it is easily seen that $S(\mu)$ is a coset of a subgroup of Γ . In particular, if $\mu = \mu^2$ and $\mu \geq 0$, then μ is the Haar measure of a compact subgroup of G ; this last statement is true even on noncommutative compact groups [34].

V. Transformations of Fourier-Stieltjes transforms. 5.1. If \mathfrak{F} is a family of functions, defined on some set S , there are two kinds of transformations of \mathfrak{F} that are often of interest: the domain transformations and the range transformations of \mathfrak{F} .

A *domain transformation* of \mathfrak{F} is a mapping α of S into S , such that $f \circ \alpha \in \mathfrak{F}$ whenever $f \in \mathfrak{F}$; here $f \circ \alpha$ is the function defined by $(f \circ \alpha)(x) = f(\alpha(x))$, for $x \in S$. In Chapter III, domain transformations of Fourier transforms and of F. S. transforms were discussed. If \mathfrak{F} is a linear space, domain transformations of \mathfrak{F} are linear operations on \mathfrak{F} .

A function Φ , defined on a set E in the complex plane, may be called a *range transformation* of \mathfrak{F} if $\Phi \circ f \in \mathfrak{F}$ for every $f \in \mathfrak{F}$ whose range lies in E . For brevity, we shall use the phrase " Φ operates in \mathfrak{F} " to describe this situation. For example, if \mathfrak{F} is an algebra (with pointwise multiplication) and if $\Phi(z)$ is a polynomial in z , then Φ operates in \mathfrak{F} (no matter what E is).

We shall be concerned with the problem of determining those Φ which operate in algebras of Fourier transforms and of F. S. transforms.

5.2. **DEFINITION OF A , F , AND FS .** We let A denote the set of all functions on C which are sums of absolutely convergent trigonometric series; F is the set of all sequences of Fourier coefficients (i.e., F is the set of Fourier transforms of the functions in $L^1(C)$); and FS is the set of all sequences of F. S. coefficients (i.e., the set of F. S. transforms of measures in $M(C)$).

5.3. The earliest theorem in this circle of ideas is due to Wiener and Lévy [35; 20]: If $f \in A$ and if Φ is analytic on the range of f , then $\Phi \circ f \in A$.

It is natural to ask whether this sufficient condition of analyticity can be weakened, but the search for strong necessary conditions has been successful only quite recently.

Let us assume that Φ is defined on the closed interval $I = [-1, 1]$. In 1954, Kahane [14; 15] proved that Φ does not operate in A if $\Phi(x) = |x|$, and more or less simultaneously I proved the same for $\Phi(x) = (1+x)^{1/2}$, and for some similar functions [22]. In 1956, I

proved that if Φ operates in F , then Φ must satisfy a Lipschitz condition of order 1 in some neighborhood of the origin; if Φ operates in FS , then the Lipschitz condition holds uniformly on I [24]. Kahane [16] countered by proving that this necessary condition was not sufficient: he showed that Φ does not operate in F if $\Phi(x) = |x|$.

Last spring a bigger step was made: Kahane [17] constructed infinitely differentiable functions which do not operate in A . But the final results were only obtained within the last month:

5.4. THEOREM. *Suppose Φ is a complex function defined on the interval $I = [-1, 1]$.*

(a) *Φ operates in A if and only if Φ is analytic at every point of I [19].*

(b) *Φ operates in F if and only if Φ is analytic in some neighborhood of the origin [11].*

(c) *Φ operates in FS if and only if Φ can be extended to an entire function in the complex plane [18].*

5.5. If Φ is defined in the whole plane (or in a region), analogous conclusions are obtained, with "real-analytic" in place of "analytic." But 5.4(c) also has the following corollary [18].

Suppose Φ is defined in the whole complex plane. Then Φ operates in the algebra of the Gelfand transforms of the measures in $M(C)$ if and only if Φ is an entire function of a complex variable.

This is a much stronger result than Theorem 2.4(a), where the function $\Phi(z) = \bar{z}$ is ruled out. However, it should be recorded that Theorem 2.4(a) led to the conjecture that 5.4(c) might be true.

5.6. I would like to end with a brief sketch of the proof of Theorem 5.4(c). For $\mu \in M(C)$, let $\Phi \circ \mu$ denote the measure σ such that $\hat{\sigma}(n) = \Phi(\hat{\mu}(n))$ for $n = 0, \pm 1, \pm 2, \dots$. One first proves that to each $r > 0$ there is an $M(r) < \infty$ such that $\|\Phi \circ \mu\| \leq M(r)$ whenever $\|\mu\| \leq r$ and $\hat{\mu}(n) \in I$, and one defines

$$(5.6.1) \quad \Psi(x) = \Phi(R \sin x) = \sum_{-\infty}^{\infty} c_n e^{inx} \quad (-\infty < x < \infty)$$

for some R satisfying $0 < R < 1$. Then for every real number a ,

$$(5.6.2) \quad \|\Psi \circ (\mu + a)\| \leq M(Re^r) = M$$

whenever $\|\mu\| \leq r$ and $\hat{\mu}$ is real on J . Since, formally,

$$(5.6.3) \quad \Psi \circ (\mu + a) = \sum_{-\infty}^{\infty} c_n e^{in\mu} e^{ina},$$

it is reasonable to expect that

$$(5.6.4) \quad \|c_n e^{in\mu}\| = \left\| \frac{1}{2\pi} \int_{-\pi}^{\pi} \Psi \circ (\mu + a) e^{-ina} da \right\| \leq M$$

$$(n = 0, \pm 1, \pm 2, \dots),$$

if $\|\mu\| \leq r$ and μ is real on J . This is not hard to prove, and if we take μ as in Lemma 2.3, we can conclude that

$$(5.6.5) \quad |c_n| e^{|n|r} \leq M \quad (n = 0, \pm 1, \pm 2, \dots).$$

Consequently the Fourier series in (5.6.1) converges to an analytic function in the strip $|y| < r$. This is true for all r , so that Ψ is an entire function.

Let us do this for two different values of R , say R_1 and R_2 , and call the corresponding entire functions Ψ_1 and Ψ_2 . We then have, in some neighborhood of the origin,

$$(5.6.6) \quad \Phi(x) = \Psi_1\left(\arcsin \frac{x}{R_1}\right) = \Psi_2\left(\arcsin \frac{x}{R_2}\right).$$

The first representation of Φ shows that R_1 , $-R_1$, and ∞ are the only possible singular points of Φ ; the second representation shows that R_2 , $-R_2$, and ∞ are the only possibilities. Since $R_1 \neq R_2$, Φ can be extended to an entire function.

Postscript (added February 4, 1959). Since this address was delivered, the following new developments have occurred in the subject under consideration:

(1) P. J. Cohen has characterized all idempotents in $M(G)$; the conjecture 4.5 is correct.

(2) Using this result, he has also characterized all homomorphisms of $L^1(G_1)$ into $M(G_2)$. A summary of this work will appear shortly as a Research Announcement in this Bulletin.

(3) One can deduce from Cohen's work that Theorem 3.5 is true without any countability restriction on Γ_1 .

(4) Theorem 5.4 has been extended to locally compact abelian groups in general. Three cases have to be distinguished (we assume $\Phi(0) = 0$, Φ is defined on I , and we let $A(\Gamma)$, $B(\Gamma)$ denote the algebras of all Fourier transforms and F. S. transforms, respectively, on Γ):

(a) If Γ is not discrete, Φ operates in $A(\Gamma)$ if and only if Φ is analytic on I .

(b) If Γ is discrete, Φ operates in $A(\Gamma)$ if and only if Φ is analytic in some neighborhood of the origin.

(c) If Γ is not compact, Φ operates in $B(\Gamma)$ if and only if Φ can be extended to an entire function in the complex plane.

Proofs of these results are being published jointly by the authors of [11; 18; 19].

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