SOME PROPOSITIONS EQUIVALENT TO THE CONTINUUM HYPOTHESIS

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Let $\mathcal{E}$ denote the real line. If $T \subset \mathcal{E}$ and $r \in \mathcal{E}$, we set $\{t+r : t \in T\} = T[r]$. In [1] we have proved these two theorems:

(B_K) *Let $S \subset \mathcal{E}$, $T \subset \mathcal{E}$, S be at most enumerable and T be of first category. Then $\mathcal{E}$ contains a residual subset R such that $S \cap T[r]$ is empty for every $r \in R$.*

(B_M) *Let $S \subset \mathcal{E}$, $T \subset \mathcal{E}$, S be at most enumerable and T be of measure zero. Then $\mathcal{E}$ contains a subset R such that $\mathcal{E} - R$ is of measure zero and $S \cap T[r]$ is empty for every $r \in R$.*

We introduce the following propositions:

($\mathfrak{B}_K$) *Let $S \subset \mathcal{E}$, $T \subset \mathcal{E}$, S be of power less than $2^{\aleph_0}$ and T be of first category. Then $\mathcal{E}$ contains a residual subset R such that $S \cap T[r]$ is empty for every $r \in R$.*

($\mathfrak{B}_M$) *Let $S \subset \mathcal{E}$, $T \subset \mathcal{E}$, S be of power less than $2^{\aleph_0}$ and T be of measure zero. Then $\mathcal{E}$ contains a subset R such that $\mathcal{E} - R$ is of measure zero and $S \cap T[r]$ is empty for every $r \in R$.*

($\mathfrak{B}_K^*$) *Let $S \subset \mathcal{E}$, $T \subset \mathcal{E}$, S be of power less than $2^{\aleph_0}$ and T be of first category. Then there exists an $r \in \mathcal{E}$ such that $S \cap T[r]$ is empty.*

($\mathfrak{B}_M^*$) *Let $S \subset \mathcal{E}$, $T \subset \mathcal{E}$, S be of power less than $2^{\aleph_0}$ and T be of measure zero. Then there exists an $r \in \mathcal{E}$ such that $S \cap T[r]$ is empty.*

Clearly ($\mathfrak{B}_K$) implies ($\mathfrak{B}_K^*$) and ($\mathfrak{B}_M$) implies ($\mathfrak{B}_M^*$).

The following five propositions are discussed at some length in [2]:

(H) $2^{\aleph_0} = \aleph_1$.

($\mathfrak{K}$) *The union of less than $2^{\aleph_0}$ subsets of $\mathcal{E}$ of first category is of first category.*

($\mathfrak{M}$) *The union of less than $2^{\aleph_0}$ subsets of $\mathcal{E}$ of measure zero is of measure zero.*

($\mathfrak{K}^*$) *$\mathcal{E}$ is not the union of less than $2^{\aleph_0}$ subsets of $\mathcal{E}$ of first category.*

($\mathfrak{M}^*$) *$\mathcal{E}$ is not the union of less than $2^{\aleph_0}$ subsets of $\mathcal{E}$ of measure zero.*

Evidently (H) implies ($\mathfrak{K}$) and ($\mathfrak{M}$), ($\mathfrak{K}$) implies ($\mathfrak{K}^*$), and ($\mathfrak{M}$) implies ($\mathfrak{M}^*$).

By examining the proofs of (B_K) and (B_M), it is easy to see that the following lemma is true.

LEMMA 1. ($\mathfrak{K}$) *implies* ($\mathfrak{B}_K$), ($\mathfrak{M}$) *implies* ($\mathfrak{B}_M$), ($\mathfrak{K}^*$) *implies* ($\mathfrak{B}_K^*$), *and* ($\mathfrak{M}^*$) *implies* ($\mathfrak{B}_M^*$).

Now let $\mathcal{P}$ denote the plane provided with a Cartesian coordinate system having a horizontal x-axis and a vertical y-axis. If Φ is a family of horizontal lines (in $\mathcal{P}$), we say that Φ is of first category (measure zero) if the union of the members of Φ intersects the y-axis in a linear set of first category (measure zero). If $r \in \mathcal{E}$, we denote by $\Phi[r]$ the family of horizontal lines obtained from Φ as follows: if L is a member of Φ and intersects the y-axis at y_0, then the horizontal line that intersects the y-axis at y_0+r is made a member of $\Phi[r]$. We call the families $\Phi[r]$ $(r \in \mathcal{E})$ the translations of Φ.

We introduce also the following propositions:

($\mathfrak{Q}_K$) *There exists a subset A of $\mathcal{P}$ and a family Φ of horizontal lines such that*

(i) *Φ is of first category,*

(ii) *there is a subset U of $\mathcal{E}$ of second category such that, for every $u \in U$, the family $\Phi[u]$ contains a horizontal line that intersects A in at most $\aleph_0$ points,*

(iii) *every member of some nonenumerable set of vertical lines intersects $\mathcal{P}-A$ in at most $\aleph_0$ points.*

($\mathfrak{Q}_M$) *There exists a subset A of $\mathcal{P}$ and a family Φ of horizontal lines such that*

(i) *Φ is of measure zero,*

(ii) *there is a subset U of $\mathcal{E}$ of positive exterior measure such that, for every $u \in U$, the family $\Phi[u]$ contains a horizontal line that intersects A in at most $\aleph_0$ points,*

(iii) *every member of some nonenumerable set of vertical lines intersects $\mathcal{P}-A$ in at most $\aleph_0$ points.*

($\mathfrak{Q}_K^*$) *There exists a subset A of $\mathcal{P}$ and a family Φ of horizontal lines such that*

(i) *Φ is of first category,*

(ii) *every translation of Φ contains a horizontal line that intersects A in at most $\aleph_0$ points,*

(iii) *every member of some nonenumerable set of vertical lines intersects $\mathcal{P}-A$ in at most $\aleph_0$ points.*

($\mathfrak{Q}_M^*$) *There exists a subset A of $\mathcal{P}$ and a family Φ of horizontal lines such that*

(i) *Φ is of measure zero,*

(ii) *every translation of Φ contains a horizontal line that intersects A in at most $\aleph_0$ points,*

(iii) *every member of some nonenumerable set of vertical lines intersects $\mathcal{P}-A$ in at most $\aleph_0$ points.*

($\mathfrak{Q}_K'$) *There exists a subset A of $\mathcal{P}$ and a family Φ of horizontal lines such that*

(i) *Φ is of power less than $2^{\aleph_0}$,*

(ii) *there is a subset U of $\mathcal{E}$ of second category such that, for every $u \in U$, the family $\Phi[u]$ contains a horizontal line that intersects A in at most $\aleph_0$ points,*

(iii) *every member of some nonenumerable set of vertical lines intersects $\mathcal{P}-A$ in a linear set of first category.*

($\mathfrak{Q}'_M$) *There exists a subset A of $\mathcal{P}$ and a family Φ of horizontal lines such that*

(i) *Φ is of power less than $2^{\aleph_0}$,*

(ii) *there is a subset U of E of positive exterior measure such that, for every $u \in U$, the family $\Phi[u]$ contains a horizontal line that intersects A in at most $\aleph_0$ points,*

(iii) *every member of some nonenumerable set of vertical lines intersects $\mathcal{P}-A$ in a linear set of measure zero.*

Obviously ($\mathfrak{Q}^*_K$) implies ($\mathfrak{Q}_K$) and ($\mathfrak{Q}^*_M$) implies ($\mathfrak{Q}_M$). We remark that Propositions ($\mathfrak{B}$) and ($\mathfrak{W}$) in [2] imply ($\mathfrak{Q}'_K$) and ($\mathfrak{Q}'_M$), respectively.

Lemma 2. (H) *implies* ($\mathfrak{Q}^*_K$), ($\mathfrak{Q}^*_M$), ($\mathfrak{Q}'_K$), *and* ($\mathfrak{Q}'_M$).

Proof. Suppose that (H) is true. Then [3, p. 9, Proposition P_1] there exists a subset A of $\mathcal{P}$ such that the intersection of every horizontal line with A is an at most enumerable set and the intersection of every vertical line with $\mathcal{P}-A$ is an at most enumerable set; if we let Φ consist of a single horizontal line, the truth of ($\mathfrak{Q}^*_K$), ($\mathfrak{Q}^*_M$), ($\mathfrak{Q}'_K$), and ($\mathfrak{Q}'_M$) is apparent.

Theorem 1. *The conjunction of* ($\mathfrak{B}_K$) *and* ($\mathfrak{Q}_K$) *is equivalent to* (H).

Proof. (a) Assume that (H) is true. Then Lemma 1 implies that ($\mathfrak{B}_K$) is true, and the truth of ($\mathfrak{Q}_K$) follows from Lemma 2.

(b) Assume that ($\mathfrak{B}_K$) and ($\mathfrak{Q}_K$) are true. If (H) is false, then, in view of (iii) of ($\mathfrak{Q}_K$), there exist $\mathfrak{p}$ vertical lines, with $\aleph_0 < \mathfrak{p} < 2^{\aleph_0}$, whose union intersects $\mathcal{P}-A$ in a set whose orthogonal projection, S, on the y-axis is of power less than $2^{\aleph_0}$. If T is the intersection of the y-axis with the union of the members of Φ, then, by (i) of ($\mathfrak{Q}_K$), T is a linear set of first category, and ($\mathfrak{B}_K$) implies that $\mathcal{E}$ contains a residual subset R with the property that $S \cap T[r]$ is empty for every $r \in R$. This means that, for some $u \in U$, every member of $\Phi[u]$ intersects each of the aforementioned $\mathfrak{p}$ vertical lines in a point of A, which contradicts (ii) of ($\mathfrak{Q}_K$). Consequently, (H) is true.

Theorem 2. *The conjunction of* ($\mathfrak{B}_M$) *and* ($\mathfrak{Q}_M$) *is equivalent to* (H).

Proof. In the proof of Theorem 1, replace "($\mathfrak{B}_K$)" by "($\mathfrak{B}_M$)",

"$(\mathfrak{Q}_K)$" by "$(\mathfrak{Q}_M)$", "first category" by "measure zero," and "residual subset R" by "subset R such that $\mathcal{E}-R$ is of measure zero."

THEOREM 3. *The conjunction of* $(\mathfrak{B}_K^*)$ *and* $(\mathfrak{Q}_K^*)$ *is equivalent to* (H).

PROOF. (a) Assume that (H) is true. Then Lemma 1 implies that $(\mathfrak{B}_K^*)$ is true, and the truth of $(\mathfrak{Q}_K^*)$ follows from Lemma 2.

(b) Assume that $(\mathfrak{B}_K^*)$ and $(\mathfrak{Q}_K^*)$ are true. If (H) is false, then, in view of (iii) of $(\mathfrak{Q}_K^*)$, there exist $\mathfrak{p}$ vertical lines, with $\aleph_0<\mathfrak{p}<2^{\aleph_0}$, whose union intersects $\mathcal{P}-A$ in a set whose orthogonal projection, S, on the y-axis is of power less than $2^{\aleph_0}$. If T is the intersection of the y-axis with the union of the members of Φ, then, by (i) of $(\mathfrak{Q}_K^*)$, T is a linear set of first category, and $(\mathfrak{B}_K^*)$ implies the existence of an $r\in\mathcal{E}$ such that $S\cap T[r]$ is empty. This means that every member of some translation of Φ intersects each of the aforementioned $\mathfrak{p}$ vertical lines in a point of A, which contradicts (ii) of $(\mathfrak{Q}_K^*)$. Consequently, (H) is true.

THEOREM 4. *The conjunction of* $(\mathfrak{B}_M^*)$ *and* $(\mathfrak{Q}_M^*)$ *is equivalent to* (H).

PROOF. In the proof of Theorem 3, replace "$(\mathfrak{B}_K^*)$" by "$(\mathfrak{B}_M^*)$", "$(\mathfrak{Q}_K^*)$" by "$(\mathfrak{Q}_M^*)$", and "first category" by "measure zero."

THEOREM 5. *The conjunction of* $(\mathfrak{K})$ *and* $(\mathfrak{Q}_K')$ *is equivalent to* (H).

PROOF. (a) Assume that (H) is true. Then, as we have remarked above, $(\mathfrak{K})$ is true, and the truth of $(\mathfrak{Q}_K')$ follows from Lemma 2.

(b) Assume that $(\mathfrak{K})$ and $(\mathfrak{Q}_K')$ are true. If (H) is false, then, in view of (iii) of $(\mathfrak{Q}_K')$, $(\mathfrak{K})$ implies that there exist $\mathfrak{p}$ vertical lines, with $\aleph_0<\mathfrak{p}<2^{\aleph_0}$, whose union intersects $\mathcal{P}-A$ in a set whose orthogonal projection, T, on the y-axis is a linear set of first category. If S is the intersection of the y-axis with the union of the members of Φ, then, by (i) of $(\mathfrak{Q}_K')$, S is of power less than $2^{\aleph_0}$, and $(\mathfrak{B}_K)$, which follows from $(\mathfrak{K})$ according to Lemma 1, implies that $\mathcal{E}$ contains a residual subset R with the property that $T\cap S[r]$ is empty for every $r\in R$. This means that, for some $u\in U$, every member of $\Phi[u]$ intersects each of the aforementioned $\mathfrak{p}$ vertical lines in a point of A, which contradicts (ii) of $(\mathfrak{Q}_K')$. Consequently, (H) is true.

THEOREM 6. *The conjunction of* $(\mathfrak{M})$ *and* $(\mathfrak{Q}_M')$ *is equivalent to* (H).

PROOF. In the proof of Theorem 5, replace "$(\mathfrak{K})$" by "$(\mathfrak{M})$", "$(\mathfrak{Q}_K')$" by "$(\mathfrak{Q}_M')$", "first category" by "measure zero," "$(\mathfrak{B}_K)$" by "$(\mathfrak{B}_M)$", and "residual subset R" by "subset R such that $\mathcal{E}-R$ is of measure zero."

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