

THE NOVEMBER MEETING IN LOS ANGELES

The five hundred eighteenth meeting of the American Mathematical Society was held at the University of Southern California, Los Angeles, California, on Saturday, November 12, 1955. Attendance was approximately one hundred, including 88 members of the Society.

By invitation of the Committee to Select Hour Speakers for Far Western Sectional Meetings, Professor Paul Garabedian delivered an address on *The mathematical theory of three-dimensional cavities and jets*. He was introduced by Professor J. W. Green. Professors R. S. Phillips and L. J. Paige presided at the sessions for contributed papers.

Following are the abstracts of papers presented at the meeting, those whose numbers are followed by "t" having been given by title. Mr. Steger was introduced by Professor A. V. Martin, Dr. Nitsche by Professor Stefan Bergman, and Dr. Banaschewski by Professor D. B. Summer.

ALGEBRA AND THEORY OF NUMBERS

40. D. L. Boyer: *Enumeration theorems in infinite Abelian groups.*

W. R. Scott [Amer. J. Math. vol. 74 (1952) pp. 187-197] has proved that an Abelian group of order $A > K_0$ has 2^A subgroups of order A and the intersection of all the subgroups is the identity. He has proved in the same paper that the intersection of all the infinite subgroups of a countable Abelian group G is the identity unless $G = Z(p^\infty) \oplus F$, where F is finite. The present paper extends the remaining parts of the above theorem to include countable Abelian groups. It is also pointed out that the above theorem is valid for modules over a principal ideal ring provided the order of the ring is less than the order of the module. Finally it is shown that the order of the automorphism group of a countable torsion Abelian group is 2^{K_0} . (Received September 26, 1955.)

41. D. H. Lehmer: *On the roots of the Riemann zeta function.*

Calculations are described which show that the first 10,000 roots of the Riemann zeta function in the upper half plane have real parts of $1/2$. The calculations were done on the SWAC in a few hours time. The basic computation consisted in evaluating a real multiple $[\exp(\pi i \theta)] \zeta(s)$ of the zeta function at the first 10,000 Gram points. Later steps were taken to dispose of doubtful situations by a refined mesh. Use was made of a suitably modified formula of Titchmarsh. Finally two near misses of the Riemann Hypothesis had to be examined by the Euler-Maclaurin formula using 2000 terms of $\sum n^{-s}$ (35 seconds) to obtain a small error. Two pairs of nearly coincident roots with serial numbers 4763, 4764, 6707, 6708 were computed. The latter pair have imaginary parts of 7005.0629 and 7005.1006. The zeta function exhibits a disregard for Gram's Law which increases with t . Over 800 violations were found. In five cases there are three roots between two consecutive Gram points. (Received October 6, 1955.)

42. T. S. Motzkin: *Simple and equi-element finite semigroups*. Preliminary report.

A finite semigroup is equi-element (i.e., has a transitive automorphism group) only if it is the direct product of a factor in which $ab=a$, for any elements a and b , and a factor in which $cd=d$ for all c and d . Simple semigroups (with no nontrivial homomorphism) of order n , besides the simple groups, exist certainly for $n=rs+1$, $r \leq s < 2r$; the number of nonisomorphic simple semigroups of a given order is unbounded. For $n=3, 4, 5$ there are no other simple semigroups, as the list mentioned in the following abstract shows. (Received November 7, 1955.)

43. T. S. Motzkin and J. L. Selfridge (p): *Semigroups of order five*.

After theoretical exclusion of impossible first rows, the lexicographically arranged multiplication tables of all nonisomorphic semigroups of order 5 and (relying on the first result of the preceding abstract) their automorphism groups have been listed, using the electronic computer SWAC. There exist 183732 semigroups of order 5, of which 1915 are nonisomorphic. Of these, 325 are commutative and 80 others are selfdual (antiisomorphic to themselves). If antiisomorphic semigroups are identified the number of types is thus 1160. (Received November 7, 1955.)

44t. R. S. Pierce: *Distributivity in Boolean algebras*. Preliminary report.

Let α and β be infinite cardinals. An α -complete Boolean algebra B is called (α, β) -distributive if the identity: (*) $\prod_{\sigma \in S} (\sum_{\tau \in T} a_{\sigma\tau}) = \sum_{\phi \in F} (\prod_{\sigma \in S} a_{\sigma\phi(\sigma)})$, where $F=T^S$, holds in B whenever $\bar{S} \leq \alpha$ and $\bar{T} \leq \beta$. B is strongly (α, β) -distributive if the above equality prevails and $\prod_{\sigma \in S} a_{\sigma\phi(\sigma)} = 0$ for all $\phi \in F$ except a subset of cardinality $\leq \alpha$. Finally, B is weakly (α, β) -distributive if (*) holds in the lattice of dual ideals of B , whenever the $a_{\sigma\tau}$ are principal dual ideals and $\sum_{\tau \in T} a_{\sigma\tau}$ is the unit ideal for all $\sigma \in S$. Sample results are: B is weakly (α, β) -distributive for all α iff B is (isomorphic to) a β -field of sets; B is weakly (α, α) -distributive iff there is an α -field F , an α -homomorphism $h: F \rightarrow B$ and a homomorphism $i: B \rightarrow F$ such that $h \circ i$ is the identity map of B ; B is (α, α) -distributive iff B is isomorphic to F/I , where F is an α -field and I is an α -ideal closed under certain 2^α -joins (if F is 2^α -complete, I must be a 2^α -ideal); B is strongly (α, α) -distributive iff B is the limit of an α -directed system of algebras isomorphic to 2^X , where $\bar{X} \leq \alpha$; B is (\aleph_0, \aleph_0) -distributive iff every real-valued, continuous function on the Boolean space of B is locally constant on a dense subset of the space. (Received October 4, 1955.)

45t. R. S. Pierce: *Products of Boolean algebras*. Preliminary report.

Let α be an infinite cardinal. An algebraic definition is given for the α -direct product $(\alpha) \prod_{\sigma \in S} B_\sigma$ of a set of Boolean algebras B . It is shown that the definition is equivalent to one given by Sikorski (Fund. Math. vol. 39 (1952)) in the case $\alpha = \aleph_0$. For each $\sigma \in S$, there is an α -isomorphism $i_\sigma: B_\sigma \rightarrow (\alpha) \prod_{\sigma \in S} B$. Main theorem: If α is a regular cardinal and $h_\sigma: B_\sigma \rightarrow B$ is an α -homomorphism, all B_σ are strongly (α, α) -distributive and B is weakly (α, α) -distributive, then there is a unique α -homomorphism $h: (\alpha) \prod_{\sigma \in S} B_\sigma \rightarrow B$ such that $h \circ i_\sigma = h_\sigma$. Corollary: The free weakly (α, α) -distributive Boolean algebra with β generators is the α -product of β replicas of the four element Boolean algebra. (Received October 4, 1955.)

46. Arthur Steger: *Direct decomposition of finitely idempotent rings*.

A. L. Foster showed that if R is a commutative ring with unit and if R^* is the set of idempotents in R then, under a suitable addition and ordinary ring multiplication, R^* is a Boolean ring. Hence, if R^* is finite it contains exactly 2^t elements. If $t=1$, R will be called minimally idempotent. The main result of this paper is the following: If R^* is finite with 2^t elements then R is the direct sum of t minimally idempotent rings. Since a commutative ring with unit is directly indecomposable if and only if it is minimally idempotent, it follows that the direct summands are uniquely determined. This theorem yields as special cases the result of McCoy and Montgomery on the decomposition of finite p -rings and the result of Harary on the decomposition of finite Boolean-like rings. The noncommutative case (with unit) yields to the same approach provided the role played by R^* is now assumed by the set of idempotents in the center of R . (Received September 26, 1955.)

47. Morgan Ward: *Linear divisibility sequences of order three.*

The form of all integral cubic linear divisibility sequences is determined where characteristic polynomial does not have three distinct integral roots. With trivial exceptions, every such sequence is the square of a Lehmer sequence. (Received October 3, 1955.)

ANALYSIS

48. M. G. Arsove: *Proper bases and automorphisms in the space of entire functions.*

A sequence $\{\alpha_n\}_{n=0}^{\infty}$ in the space Γ of entire functions is a *basis* if every $f \in \Gamma$ can be represented uniquely as $f = \sum_{n=0}^{\infty} c_n \alpha_n$, where the c_n 's are complex constants and the series converges uniformly on compact sets. If, further, $\sum_{n=0}^{\infty} |c_n \alpha_n|$ converges uniformly on compact sets, the basis is *absolute*. A basis in which $|c_n|^{1/n} \rightarrow 0$ for all f is (Iyer)-*proper*, i.e. proper in the sense of Iyer [Proc. Amer. Math. Soc. vol. 3 (1952) pp. 874-883]. Iyer's conjecture that all bases are (Iyer)-proper is observed to be false (e.g. $\alpha_n(z) = z^n/n!$). Modifying Iyer's terminology, the author calls $\{\alpha_n\}$ a *proper basis* provided it is an absolute basis and $\sum_{n=0}^{\infty} |c_n \alpha_n|$ converges uniformly on compact sets if and only if $|c_n|^{1/n} \rightarrow 0$. Proper bases are characterized in terms of $M_n(R) = \max_{|z|=R} |\alpha_n(z)|$ as those absolute bases for which $\limsup_{n \rightarrow \infty} [M_n(R)]^{1/n} < +\infty$ for each $R > 0$ and $\lim_{R \rightarrow \infty} \{\liminf_{n \rightarrow \infty} [M_n(R)]^{1/n}\} = +\infty$. Theorem: the linearly homeomorphic images of Γ in itself coincide with the closed subspaces Γ_0 admitting proper bases. For the case of automorphisms ($\Gamma_0 = \Gamma$) this resolves certain questions left open by Iyer. It is shown how the successive remainders of suitably restricted functions of exponential type generate proper bases, and inversion yields a class of proper bases in which each α_n is a polynomial of degree $n+1$. (Received September 30, 1955.)

49. W. G. Bade (p) and J. T. Schwartz: *On abstract eigenfunction expansions.*

Let T be a self-adjoint operator in $L_2(S, \Sigma, \nu)$ where (S, Σ, ν) is a positive measure space. Let E be the resolution of the identity for T . Assume there exists an increasing sequence $\{S_n\}$ of sets covering S , $\nu(S_n) < \infty$, such that for each bounded Borel set e and $f \in L_2(S, \Sigma, \nu)$, ν -ess. sup. $|(E(e)f)(s)| < \infty$ on S_n , $n=1, 2, \dots$. Then T has an abstract eigenfunction expansion in the sense of Mautner (Proc. Nat. Acad. Sci. U.S.A. vol. 39 (1953) pp. 49-53); i.e., there exist Borel measures μ_α on the real line and functions $W_\alpha(s, \lambda)$ defined and measurable with respect to $\nu \times \mu_\alpha$ on $S \times \text{reals}$ and a linear isometry U of $L_2(S, \Sigma, \nu)$ onto $\Sigma_\alpha L_2(\mu_\alpha)$ which diagonalizes T such that

$(Uf)_\alpha(\lambda) = \int_S f(s) \overline{W_\alpha(s, \lambda)} \nu(ds)$ and $f(\cdot) = \sum_\alpha \int_{-\infty}^{\infty} (Uf)_\alpha(\lambda) W_\alpha(\cdot, \lambda) \mu_\alpha(d\lambda)$, the integrals existing in the mean square sense. Moreover, ν -ess. sup. $\int_e |W_\alpha(s, \lambda)|^2 \mu_\alpha(d\lambda) < \infty$ on S_n for each bounded Borel set e and $n = 1, 2, \dots$. If $\mu_{\alpha_1} \geq \mu_{\alpha_2} \geq \dots$, the functions $W_{\alpha_1}(\cdot, \lambda), \dots, W_{\alpha_n}(\cdot, \lambda)$ are linearly independent for μ_{α_n} -almost all λ . The results may be applied to simplify recent work of Gårding and Browder on eigenfunction expansions for elliptic operators. If T is an ordinary differential operator of order n , the spectral multiplicity of T is $\leq n$. The kernels $W(s, \lambda)$ satisfy the boundary conditions at any fixed end point. (Received October 7, 1955.)

50. Seymour Ginsburg: *On mappings from the family of well ordered subsets of a set.*

For a simply ordered set E let ωE denote the family of all nonempty well ordered subsets of E ordered as follows: $W_1 <_k W_2$ if W_1 is a proper initial segment of W_2 . A function f from ωE to E is called a k -function if $W_1 <_k W_2$ implies $f(W_1) < f(W_2)$. If there exists a k -function on E , then E is called a k -set. The existence of k -sets is studied. For example, each simply ordered set E is similar to a terminal segment of some k -set $F(E)$. It is not true that each simply ordered set E is similar to an initial segment of some k -set $F(E)$. Finally it is shown that no infinite, simply ordered group is a k -set. (Received September 26, 1955.)

51. C. J. A. Halberg, Jr.: *Remarks concerning the spectra of bounded linear operators on the sequence spaces.* Preliminary report.

Let $[l_p]$ denote the algebra of all bounded linear operators mapping the sequence space l_p into itself. Given an infinite matrix A , let A_p and A_p^t denote the operators defined on l_p by the matrix and its transpose respectively, and let $\|A_p\|$ denote the norm of A_p as an operator on l_p . By the use of a convexity theorem of M. Riesz the validity of the inequality $\|A_{1/\alpha_2}\|^{(\alpha_3-\alpha_1)} \leq \|A_{1/\alpha_1}\|^{(\alpha_3-\alpha_2)} \|A_{1/\alpha_3}\|^{(\alpha_2-\alpha_1)}$ for $0 < \alpha_1 < \alpha_2 < \alpha_3$ is established. Let $\sigma(A_p)$ and $|\sigma(A_p)|$ denote the spectrum and the spectral radius, respectively, of the operator A_p . The above inequality is used to derive some results of the following types: *Theorem 1.* If $A_{1/\alpha} \in [l_{1/\alpha}]$, then $\log |\sigma(A_{1/\alpha})|$ is a convex function of α for $0 \leq \alpha \leq 1$. *Theorem 2.* If both T_p and T_p^t belong to $[l_p]$, $1 \leq p \leq 2$, and $|\sigma(T_p)| \neq |\sigma(T_p^t)|$, then either $|\sigma(T_p)| > |\sigma(T_2)|$ or $|\sigma(T_p^t)| > |\sigma(T_2^t)|$, according as $|\sigma(T_p)| > |\sigma(T_p^t)|$ or $|\sigma(T_p^t)| > |\sigma(T_p)|$. *Theorem 3.* Suppose that both T_p and T_p^t belong to $[l_p]$ and $1 \leq p \leq q \leq 2$. Then (a) $\sigma(T_q) \subset \sigma(T_p) \cup \sigma(T_p^t)$; (b) if C is any component of $\sigma(T_q)$, then the set $C \cap (\sigma(T_p) \cap \sigma(T_p^t))$ is nonvoid. (Received October 3, 1955.)

52t. D. G. Kendall: *Some further pathological examples in the theory of denumerable Markov processes.*

(i) An outstanding uniqueness problem concerning the Kolmogorov differential equations is answered in the negative by the construction of two distinct honest Markov processes (transition semigroups of operators on l) having the same (finite) values for the limits $q_{ij} \equiv \lim [p_{ij}(t) - \delta_{ij}]/t \downarrow (t \rightarrow 0)$, with $\sum_{j=0}^{\infty} q_{ij} = 0$, and satisfying both the "backward" and the "forward" equations. This improves an earlier result of Ledermann and Reuter; they also found two distinct solutions, but one of these was a dishonest "quasi-process" (contraction semigroup). (ii) Kolmogorov has given examples in which (a) $q_{00} = -\infty$, and (b) $\sum_{j \neq 0} q_{0j} < -q_{00}$, and he has remarked that the extreme pathology, (c) $q_{00} = -\infty$ and $q_{0j} = 0$ (all $j \neq 0$), could be realized by similar methods. Here a transition semigroup of operators on l is constructed for which the associated process has property (c). Informally one can say of the system that "it

leaves state 0 with infinite velocity to go—nowhere in particular.” (Received August 22, 1955.)

53*t*. D. G. Kendall and G. E. H. Reuter: *Some ergodic theorems for semigroups of operators.*

Let $G \equiv \{T_t; t \geq 0\}$ be a semigroup of operators on a Banach space X , strongly continuous for $t > 0$. Call $x \in X$ ergodic if $T_t x$ has some kind of generalized limit as $t \rightarrow \infty$, and call G ergodic if each $x \in X$ is ergodic. With suitable restrictions on G : (1) Abel or (C, k) summability, and weak or strong convergence, give four equivalent definitions of ergodicity. (2) G and the discrete semigroup $\{(\lambda J_\lambda)^n; n = 0, 1, 2, \dots\}$, J_λ being the resolvent operator for any one $\lambda > 0$, have similar ergodic properties. (3) Suitable compactness properties of J_λ imply strong or uniform ergodicity of G ; reflexive semigroups are strongly ergodic. (4) The subspace of ergodic elements can be calculated from the infinitesimal generator of G . (5) Ergodic properties of G for general weak topologies are studied. (6) Strong ergodicity of the adjoint semigroup of G implies strong ergodicity of G , if X is weakly sequentially complete. (Received September 27, 1955.)

54. V. A. Kramer: *Asymptotic inverse series.*

Let $H(t)$ be the operator defined on a Hilbert space by the series (possibly finite), $H(t) = H_0 + tH_1 + t^2H_2 + \dots$. Suppose that \mathcal{D} , the domain of $H(t)$, is dense, that H_0 is self-adjoint and bounded below by 1, that H_i for $i > 0$ is symmetric and bounded below by 0, and that $t \geq 0$. Then $H(t)$ has a Friedrichs extension, $\tilde{H}(t)$. Suppose that \tilde{H}_0 is the Friedrichs extension of its own contraction to \mathcal{D} . The extended operator $\tilde{H}(t)$ has a bounded inverse with the formal expansion $\tilde{H}^{-1}\phi = A_0\phi + tA_1\phi + t^2A_2\phi + \dots$, where the A_i 's are obtained by an identification of coefficients. If (for fixed N) $\phi \in \mathcal{D}(A_i)$ for $i \leq N$ and $A_i\phi \in \mathcal{D}$ for $i < N$, then $\lim_{t \rightarrow 0} t^{-N} \|\tilde{H}^{-1}\phi - \sum_{i=0}^N t^i A_i\phi\| = 0$. If in addition $A_N\phi$ belongs to $\mathcal{D}^{1/2}$, the domain of the square root of $\tilde{H}(t)$, then an error estimate can be given. If ϕ, ψ satisfy the first set of conditions above for N, M respectively, then $(\tilde{H}^{-1}\phi, \psi)$ can be expanded to order $N+M$. If in addition $A_N\phi$ and $A_M\psi$ belong to $\mathcal{D}^{1/2}$ then $(\tilde{H}^{-1}\phi, \psi)$ can be expanded to order $N+M+1$. These results have applications to perturbation theory. (Received October 3, 1955.)

55. L. Kuipers: *Note on the location of zeros of polynomials (a converse of Jensen's theorem concerning real polynomials).*

In the theory of the geometry of the zeros of polynomials $f(z)$ much attention has been paid to the problem of the location of the zeros of the derivative $f'(z)$, the zeros of $f(z)$ being known (Lucas's theorem, Jensen's theorem). The converse of the above problem, namely to give information on the location of the zeros of the primitives $\int f(z) dz$ of a given polynomial $f(z)$ is less often studied. See Chamberlin and Wolfe: *Note on a converse of Lucas's theorem*, Proc. Amer. Math. Soc. vol. 5 (1954) p. 203. Now let $\mathfrak{S}_{a,b}$ denote the orthogonal hyperbola with $a+bi$ and $a-bi$ as vertices. Then we have: In the closed interior of any $\mathfrak{S}_{a,b}$ of a real polynomial $f(z)$ vanishing for $z = a \pm bi$ lies at least one pair of conjugate complex zeros of the real primitive $\int_0^z f(\zeta) d\zeta + C$ (C real). This follows from Jensen's theorem. By repetition of the argument we have furthermore: If the real polynomial $\bar{f}(\bar{z})$ has complex zeros $a \pm bi$, then the polynomial $\int_0^z (z-t)^n f(t) dt + c_0 z^n + c_1 z^{n-1} + \dots + c_n$ (all c_i real) has in the closed interior of the hyperbola $(1/(n+1))(x-a)^2 + b^2 = y^2$ ($n = 2, 3, \dots$) a pair of conjugate zeros. Other similar results can be derived. (Received October 3, 1955.)

56t. R. M. Redheffer: *Characteristic values, zeros, and comparison.*

In a closed bounded region R let k be a characteristic value for the problem $z_{xx} + z_{yy} + kz = 0$, $z = 0$ on the boundary. If $w \in C^{(2)}$ and $w \neq 0$ in R , $-k \leq \max (w_{xx} + w_{yy})/w$. Hence, if the boundary is smooth, $-k = \min_w \max_{(x,y)} (w_{xx} + w_{yy})/w$. More generally, let $Az_{xx} + 2Bz_{xy} + Cz_{yy} + Dz = 0$, where A, B, C, D are arbitrary but well-defined functions of $(z_x/z, z_y/z, x, y)$, with $A \geq 0, C \geq 0, AC \geq B^2$. Assume z continuous in R , $z = 0$ on the boundary, $z \neq 0$. If $Aw_{xx} + 2Bw_{xy} + Cw_{yy} + D^*w = 0$ with $D^* > D$, then w vanishes somewhere in R . Similar results in one dimension yield an extension of Sturm's theorem on separation of zeros. This extension embraces nonlinear equations and the classical linear equation $z'' + a(x)z' + b(x)z = 0$ with no continuity or measurability hypothesis on a, b . (Received October 12, 1955.)

57. H. L. Royden: *Rings of meromorphic functions.*

Let D_1 and D_2 be two plane domains each having the property that for each boundary point there is a bounded analytic function in the domain with that point as an essential singularity. Kakutani and Chevalley have shown that if the rings B_1 and B_2 of bounded analytic functions on these two domains are algebraically isomorphic then the domains are conformally equivalent. In this talk the following generalization is given: Let R_1 and R_2 be any two rings of meromorphic functions on D_1 and D_2 which contain the rings B_1 and B_2 respectively. Then if R_1 and R_2 are algebraically isomorphic, the domains D_1 and D_2 are conformally equivalent. The proof is effected by characterizing algebraically the bounded functions in R_1 and R_2 and then applying the Kakutani-Chevalley theorem. (Received October 4, 1955.)

58. O. K. Smith: *Discontinuous oscillations.*

The system (*) $\dot{x} - k\dot{y} - ky = 0, G(\dot{y}) + x = 0$ ($\dot{x} = dx/dt, \dot{y} = dy/dt$) is investigated by considering the phase portrait in the cylindrical surface $G(\dot{y}) + x = 0$ in Cartesian (x, y, \dot{y}) space rather than in a phase plane. On the surface, lines where $G'(\dot{y}) = 0, G''(\dot{y}) \neq 0$ form boundaries across which trajectories cannot be extended continuously with t increasing, therefore *discontinuous solutions* are defined. It is shown that if $k > 0$ and G is subjected to appropriate restrictions, for instance if $G(\dot{y}) = v - 2 \arctan v$, then there exists a unique discontinuous periodic solution and this solution is orbitally stable. Similar results are obtained for the system (**) $\dot{x} + \dot{y} - G(\dot{y}) + x = 0, \dot{x} + \dot{y} - G(\dot{x}) + y = 0$ by considering a phase surface in (x, y, \dot{x}, \dot{y}) space. Both (*) and (**) arise from vacuum tube circuits and are discussed, without proofs, by Andronow and Chaikin in *Theory of oscillations* (Princeton University Press, 1949). (Received October 3, 1955.)

59. G. T. Thompson: *On Bateman's method for solving linear integral equations.*

The integral equations considered here are those of Fredholm type and second kind, that is, of the form $x(s) - \lambda \int_0^1 K(s, t)x(t)dt = y(s), 0 \leq s \leq 1$, where $y(s)$ and $K(s, t)$ are given and $x(s)$ is to be found. If $K(s, t)$ is a bilinear form of $2n$ functions $\phi_1(s), \dots, \phi_n(s); \Psi_1(t), \dots, \Psi_n(t)$, the equation above reduces to a system of n algebraic equations. In this case a kernel is said to be of "finite rank" or "degenerate." Ordinarily $K(s, t)$ is not of finite rank; however, one may replace $K(s, t)$ by an approximation of finite rank and solve the resulting equations. Bateman accomplished this systematically with determinantal formulas which are derived outside the framework of Fredholm's equations. In this paper Bateman's method is analyzed in detail for the

physically important case in which $K(s, t)$ is the Green's function for the self-adjoint second order linear differential operator $L(u) \equiv (pu) - qu$ with homogeneous boundary conditions. An error bound is derived which is $O(\epsilon_n/n^{1/2})$ where ϵ_n is a measure of the error incurred in the approximation to $K(s, t)$ by $K_n(s, t)$. (Received September 19, 1955.)

APPLIED MATHEMATICS

60. R. G. Stoneham: *The propagation of waves in a semi-infinite nonhomogeneous elastic solid.*

Let an inhomogeneity be introduced into an elastic half-space by assuming that in the neighborhood of the free surface $z=0$ the Lamé constants are functions of the depth. The equations of motion for such an elastic medium are derived. Typically, one has $(\lambda + \mu)\partial e/\partial z + \mu \nabla^2 w + e\partial\lambda/\partial z + 2\partial\mu/\partial z \cdot \partial w/\partial z = \rho\partial^2 w/\partial t^2$ for the z -component of the displacement. Reasonable interpretation of the expected variation of the physical constants with depth may be introduced by assumptions on the character of $\lambda(z)$ and $\mu(z)$. In this paper is presented an exact solution in the form of complex contour integrals for the displacements produced by a continuous harmonic point source disturbance. The solution is based on the method of images wherein the point source is constructed from a Hankel integral transform of a one-dimensional Green's function which has the proper discontinuity as well as Sommerfeld radiation condition. Later further work will be presented on the evaluation of the contour integrals for the displacements in order to determine whether physically sensible waves forms are exhibited which are not given by a purely elastic model. (Received September 29, 1955.)

61t. A. C. Sugar: *Definitional axiom systems for thermodynamics.*

A substantial step in the clarification of thermodynamics was taken in an earlier paper by the writer (Bull. Amer. Math. Soc. Abstract 60-6-756). Here, for the first time, an important branch of thermodynamics was axiomatized. An examination shows that all of the axioms are definitional in character. One can classify this set of axioms as being two-dimensional. In addition to analyzing the general notion of a definitional axiom system, this paper is also concerned with axiomatizing a system of one-dimensional systems of thermodynamics. An example of a one-dimensional system is a mass of water under atmospheric pressure between 0° and 100°C . Clearly such a system has only one thermodynamic coordinate. For example, by a differentiable one-dimensional classical reversible system of thermodynamics we mean an ordered quadruple, $\Sigma_1 = \langle c, T, \theta, q \rangle$, which satisfies axioms B1-B4. B1. c is a positive real number. B2. T is a closed interval of real numbers. B3. θ is a differentiable function whose domain is T and whose value is a real number. B4. $q = c\theta$. If we isolate two realizations of these one-dimensional systems at different temperatures, in juxtaposition, we obtain an irreversible two-component system. More generally, the irreversible phenomenon of calorimetry can be axiomatized in the following fashion. The ordered set $\Sigma_2 = \langle c, \sigma, \tau, m, \theta, \Delta q, s \rangle$ which satisfies axioms C1-C7 is called a calorimetric system of thermodynamics. C1. c is a positive real number. C2. σ is a set of n elements, w_1, \dots, w_n . C3. τ is a set consisting of two real numbers t_1 and t_2 . C4. m is a positive real valued function on $\sigma \times \tau$ such that $m(\sigma_i, t_1) = m(\sigma_i, t_2)$. C5. θ is a function on $\sigma \times \tau$ such that $\theta(w_i, t_2) = [1/\sum_{i=1}^n m(w_i)] \sum_{i=1}^n m(w_i)\theta_i(t_2)$, $i=1, \dots, n$. C6. Δq is a real valued sequence such that $\Delta q_i = cm(w_i) [\theta(w_i, t_2) - \theta(w_i, t_1)]$. C7. $s = m \ln \theta$. Finally, it is observed that the latter two axiom systems are also definitional. (Received October 5, 1955.)

62t. C. H. Wilcox: *An expansion theorem for electromagnetic fields.*

I.

Let D be an exterior domain bounded internally by a regular surface S . Let \mathbf{A} be of class C^2 in the closure of D and let it satisfy (1) $\nabla_x(\nabla_x \mathbf{A}) = k^2 \mathbf{A}$ in D and (2) $\lim_{R \rightarrow \infty} \int_{r=R} |r_x(\nabla_x \mathbf{A}) + ik\mathbf{A}|^2 dS = 0$ (the vector radiation condition). Here k^2 is an arbitrary nonzero complex number and $\text{Im } k \geq 0$. It is shown that (I) \mathbf{A} can be represented as an integral over S involving the values of \mathbf{A} and $\nabla_x \mathbf{A}$ on S , and (II) if S is contained in the sphere $r=c$, where (r, θ, ϕ) are spherical coordinates, then $\mathbf{A} = 1/r \cdot \exp(ikr) \sum_{n=0}^{\infty} (1/r^n) \mathbf{A}_n(\theta, \phi)$. The series converges in $r > c$ and converges absolutely and uniformly in the variables r, θ , and ϕ in any region $r \geq c + \epsilon > c$. The series can be differentiated termwise with respect to r, θ , and ϕ any number of times and the resulting series all converge absolutely and uniformly. (Received October 6, 1955.)

63t. C. H. Wilcox: *An expansion theorem for electromagnetic fields.*

II.

Let \mathbf{A} satisfy the hypotheses of the preceding abstract. It is shown that (I) $\mathbf{A}_0(\theta, \phi)$ is tangent to any sphere $r=r_0$ at the point (r_0, θ, ϕ) , and (II) \mathbf{A}_0 determines $\mathbf{A}_n = A_n^1 \hat{r} + A_n^2 \hat{\theta} + A_n^3 \hat{\phi}$ through the recursion relations $ikA_n^1 = (-1/\sin \theta) [\partial(\sin \theta A_0^1)/\partial \theta + \partial A_0^3/\partial \phi]$, $2iknA_n^1 = n(n-1)A_n^1 + DA_n^1$, and $2iknA_n^2 = n(n-1)A_n^2 + DA_n^2 + D_\theta A_{n-1}$, $2iknA_n^3 = n(n-1)A_n^3 + DA_n^3 + D_\phi A_{n-1}$ where $Df = (1/\sin \theta) \partial(\sin \theta f)/\partial \theta + (1/\sin^2 \theta) \partial^2 f/\partial \phi^2$ is Beltrami's operator for the sphere, while $D_\theta F = 2\partial F^1/\partial \theta - (1/\sin^2 \theta) F^2 - (2 \cos \theta/\sin^2 \theta) \partial F^3/\partial \phi$, $D_\phi F = (2/\sin \theta) \partial F^1/\partial \phi + (2 \cos \theta/\sin^2 \theta) \partial F^2/\partial \phi - (1/\sin^2 \theta) F^3$. A field $\mathbf{A}(r)$ that is of class C^2 and that satisfies $\nabla_x(\nabla_x \mathbf{A}) = k^2 \mathbf{A}$ and $\lim_{R \rightarrow \infty} \int_{r=R} |r_x(\nabla_x \mathbf{A}) + ik\mathbf{A}|^2 dS = 0$ in an exterior domain D is called a *vector radiation function* for D . Similarly a scalar field $u(r)$ that is of class C^2 and satisfies $\nabla^2 u + k^2 u = 0$ and $\lim_{R \rightarrow \infty} \int_{r=R} |\partial u/\partial r - ik u|^2 dS = 0$ is called a *scalar radiation function* for D . The expansion theorem is used to demonstrate the following relationships between vector and scalar radiation functions: (III) \mathbf{A} is a vector radiation function for D if and only if the Cartesian components of \mathbf{A} are scalar radiation functions for D and, where $\mathbf{r} = r\hat{r}$, (a) $\lim_{r \rightarrow \infty} \exp(\text{Im } kr) \cdot \mathbf{A}(r) = 0$; (IV) The last result also holds if (a) is replaced by (b) $\nabla \cdot \mathbf{A} = 0$ on the boundary of D . (Received October 6, 1955.)

64t. C. H. Wilcox: *On the representation of electromagnetic fields by Debye potentials.*

A vector field \mathbf{A} is a *vector wave function* for a domain V if $\mathbf{A} \in C^2$ and $\nabla_x(\nabla_x \mathbf{A}) = k^2 \mathbf{A}$ in V . Similarly, a scalar field u is a *scalar wave function* for V if $u \in C^2$ and $\nabla^2 u + k^2 u = 0$ in V . A vector (scalar) wave function $\mathbf{A}(u)$ is a *vector (scalar) radiation function* for V if V is an exterior domain and $\mathbf{A}(u)$ satisfies the vector (scalar) radiation condition (see preceding abstract). Time-harmonic electromagnetic and acoustic radiation fields are important examples of vector and scalar radiation functions, respectively. It is well known that if u is a scalar wave function and $\Pi = u\mathbf{r}$, where \mathbf{r} is a position vector, then $\nabla_x \Pi$ and $\nabla_x(\nabla_x \Pi)$ are vector wave functions. The principal results of this paper are (I) Let \mathbf{A} be a vector wave function for a domain V defined by $a < r < b$ where (r, θ, ϕ) are spherical coordinates. Then there exist functions u, v such that (1) u and v are scalar wave functions for V , (2) $\int_\Omega u(r\hat{r}) d\Omega = \int_\Omega v(r\hat{r}) d\Omega = 0$ where Ω is the unit sphere $r=1, \hat{r} \in \Omega$, and $d\Omega = \sin \theta d\theta d\phi$ is the element of area on Ω , (3) $\mathbf{A} = \nabla_x(\nabla_x \Pi) + k\nabla_x \Pi'$ in V , where $\Pi = u\mathbf{r}$, $\Pi' = v\mathbf{r}$, and u and v are uniquely determined by (1), (2), and (3). They are called the electric and magnetic Debye potentials, respectively, for \mathbf{A} . (II) u is related to the radial component $A_1 = \mathbf{A} \cdot \hat{r}$ by the reciprocal

formulas $Du = -rA_1$ where $Df = (1/\sin \theta) \partial(\sin \theta f / \partial \theta) / \partial \theta + (1/\sin^2 \theta) \partial^2 f / \partial \phi^2$ and $u(r\hat{r}) = -(r/2\pi) \int_{\Omega} (\log \sin \rho/2) A_1(r\hat{r}') d\Omega'$ where $\rho(\hat{r}, \hat{r}')$ is the geodesic distance from \hat{r} to \hat{r}' on Ω . v is similarly related to $\nabla x \mathbf{A} \cdot \hat{r}$. (III) \mathbf{A} is a vector radiation function for V if and only if the corresponding Debye potentials u and v are scalar radiation functions for V . (Received September 29, 1955.)

65. C. H. Wilcox: *On isolated singularities of electromagnetic fields.*

Time-harmonic electromagnetic fields are described by vector wave functions (defined in the preceding abstract). In this paper the behavior of such functions near isolated singularities is investigated. Results obtained include the following. (I) Let \mathbf{A} have an isolated singularity at P_0 . Then $\mathbf{A} = \mathbf{A}' + \mathbf{A}''$ where \mathbf{A}' is regular at P_0 and \mathbf{A}'' is a vector radiation function for the domain whose complement is P_0 . The radiation functions $\mathbf{H}_n = k \nabla x h_n(kr) S_n(\theta, \phi) \mathbf{r}$ and $\mathbf{E}_n = \nabla x \nabla x h_n(kr) S_n(\theta, \phi) \mathbf{r}$ are called electromagnetic multipoles of order n and of magnetic and electric type, respectively. (Here $h_n(x) = (\pi/2x)^{1/2} H_{n+1/2}^{(1)}(x)$ and $S_n(\theta, \phi)$ is a spherical harmonic of order n .) (II) Let \mathbf{A} be a radiation function for the domain $r > c \geq 0$. Then $\mathbf{A} = \mathbf{E}_1 + \mathbf{E}_2 + \dots + \mathbf{H}_1 + \mathbf{H}_2 + \dots$. The series converges in $r > c$ and converges uniformly in $r \geq c + \epsilon > c$. The isolated singularity at P_0 ($r=0$) has finite order $\mu \geq 0$ if $\mathbf{A}(P) = O(1/r^\mu)$, $r \rightarrow 0$. (III) If \mathbf{A} has an isolated singularity of order μ at $r=0$ then $\mathbf{A} = \mathbf{A}' + \mathbf{A}''$ where \mathbf{A}' is regular at $r=0$ and \mathbf{A}'' consists of a finite number of electromagnetic multipoles. The orders of those of electric (magnetic) type do not exceed $[\mu - 2]$ ($[\mu - 1]$). Simple corollaries of this result are (IV) if \mathbf{A} has order $\mu < 2$ at P_0 then the singularity at P_0 is removable and (V) if \mathbf{A} has order $\mu < 3$ at P_0 ($r=0$) then $\mathbf{A} = \mathbf{A}' + \nabla x(e^{ikr}/r) \mathbf{u}$ where \mathbf{A}' is regular at P_0 and \mathbf{u} is a constant vector. (Received September 29, 1955.)

GEOMETRY

66t. J. R. Munkres: *The equivalence problem for differentiable manifolds of dimension 2.*

The equivalence problem for differentiable manifolds is the following: "Are two topologically equivalent differentiable manifolds also equivalent in the sense of differential geometry?" The author solves this problem for manifolds of dimension 2. He proves the following approximation theorem: Let M and N be two homeomorphic differentiable manifolds of class C^1 and dimension 2; no compactness assumptions are made. Let h be a homeomorphism of M onto N ; let $\phi(x)$ be a positive continuous function defined on M ; and let $d(x, y)$ be a metric on N . Then there is a homeomorphism g of M onto N which is of class C^1 and has nonvanishing Jacobian, such that for each x in M , $d(g(x), h(x)) < \phi(x)$. The methods used involve the C^1 triangulations of J. H. C. Whitehead (Ann. of Math. vol. 41 (1940) pp. 809-824) and known facts about approximating an arbitrary homeomorphism between triangulated 2-manifolds by a piecewise-linear homeomorphism. (Received September 29, 1955.)

67. Johannes Nitsche: *A uniqueness theorem on the deformation of surfaces.*

The question of the number of different realizations in R_3 of an abstractly defined $(m+1)$ -times connected piece of a surface \mathfrak{F} is discussed. The surface is defined by its metric ds^2 (with positive Gaussian curvature) and for its boundary strip there holds a relation (*) $a(s) \cdot \rho_n + b(s) \cdot \tau_g = c(s)$. ρ_n means here the normal curvature, τ_g the geodetic torsion, and $a(s)$, $b(s)$, $c(s)$ are functions of the arc length on the boundary. The integer n defined by $2\pi i n = \oint d \log(a+ib)$ is called the index corresponding to

the condition (*). This question leads to the consideration of Gauss-Codazzi's equations, which because of the theorema egregium can be understood as a quasilinear system of elliptic differential equations. Employing results of I. N. Vekua (880 Mat. Sbornik N.S. vol. 31 (73) (1952) pp. 217-314) there can be established the theorem: Let \mathfrak{F} be a solution of the problem. (i) $n > 2(m-1)$. Then \mathfrak{F} is uniquely determined. (ii) $n \leq m-2$. Then there exists a $[3(m-1)-2n]$ -parametric set of deformations of \mathfrak{F} . For the purpose of a unique determination of \mathfrak{F} further conditions must be added. The intermediate cases $m-1 \leq n \leq 2(m-1)$ (for example, the case of a doubly-connected surface for which the spatial curvature of the boundary curve is given) have to be studied in particular (see Joachim Nitsche, Math. Zeit. vol. 62 (1955) pp. 388-401). (Received October 24, 1955.)

TOPOLOGY

68*t*. Bernhard Banaschewski: *Local connectedness of extension spaces.*

Given an extension E^* of a space E (E dense in E^*), each $u \in E^* - E$ determines on E a filter (in the sense of N. Bourbaki) consisting of all $U \cap E$, U any neighborhood of u in E^* . Many properties of E^* can be related to these so-called trace filters $\gamma(u)$. As shown elsewhere (B. Banaschewski, *Überlagerungen von Erweiterungsräumen*, to appear in Archiv der Mathematik) the following condition for filters \mathfrak{A} is of interest in this respect: (C) If $O \cup P \in \mathfrak{A}$, O and P open disjoint, then either $O \in \mathfrak{A}$ or $P \in \mathfrak{A}$. It is first proved that an extension E^* of a locally connected E for each of whose $\gamma(u)$ (C) holds is locally connected if and only if each $\gamma(u)$ has a basis consisting of connected open sets. From this and the previously obtained results (loc. cit.) that (C) holds for the maximal open, maximal regular and maximal completely regular filters is then deduced: For locally compact spaces E , denumerable at infinity, the Čech compactification βE , Alexandroff's extension $\alpha' E$, and Katětov's maximal Hausdorff extension of E can never be locally connected. (Received October 3, 1955.)

69*t*. Herbert Federer: *A study of function spaces by spectral sequences.*

Suppose Y is a simple space (i.e. the operations of the fundamental group on all the homotopy groups are trivial), X is a finite-dimensional cellular space, and $v \in Y^X$. For each cell complex K on X , with p -skeletons K^p , the restriction operation defines maps of Y^X into Y^{K^p} ; let G_n^p be the kernel of the induced homomorphism of $\pi_n(Y^X, v)$ into $\pi_n(Y^{K^p}, v|K^p)$; also let $G_n^{-1} = \pi_n(Y^X, v)$. Then for $n \geq 1$ and $p \geq 0$ the factor group G_n^{p-1}/G_n^p is isomorphic to the limit group $E_n^{p,n}$ of a spectral sequence whose term $E_2^{p,n}$ is isomorphic to the p dimensional cohomology group of X with coefficients in $\pi_{p+n}(Y)$. This spectral sequence arises from an exact couple obtained from the homotopy sequences of the fibre maps of Y^{K^p} into $Y^{K^{p-1}}$ defined by the restriction operation. While the exact couple depends on K , its derived couples are homotopy type invariants of X , Y and v . (Received October 5, 1955.)

70. V. L. Klee, Jr.: *Fixed-point sets of periodic homeomorphisms of Hilbert space.*

Considerable space in the literature has been devoted to the following question, for various metric spaces X : If Y is the set of all fixed points of a periodic homeomorphism of X , what topological properties of Y can be deduced from those of X ? In the most-studied cases, the currently known results assert (under various additional

hypotheses) that if X is $E^n [S^n]$, then Y is in some sense homologically similar to $E^k [S^k]$ for an appropriate $k < n$. The present note establishes the following result, whose contrast with the finite-dimensional situation is striking: *If Y is a compact [closed] subset of an infinite-dimensional Hilbert space H and n is an integer ≥ 2 , then H admits a homeomorphism of period n whose fixed-point set is Y [is homeomorphic with Y].* The basic tools employed are contained in earlier papers by the author [Trans. Amer. Math. Soc. vol. 74 (1953) pp. 10–43 and vol. 78 (1955) pp. 30–45]. (Received October 5, 1955.)

71t. E. A. Michael: *On a theorem of Borsuk.*

The space of nonempty, closed, LC^n subsets of a complete metric space is metrized with a complete metric in such a way that convergence of a sequence of sets in this metric is equivalent to homotopy- n -regular convergence in the sense of M. L. Curtis [Ann. of Math. vol. 57 (1953) pp. 231–247]. This slightly generalizes a result of K. Borsuk [Fund. Math. (1954) pp. 168–202] and K. Kuratowski [Bull. Acad. Polon. Sci. Cl. III vol. 3 (1955) pp. 75–80]. (Received October 5, 1955.)

72t. J. R. Munkres: *The general triangulation problem for dimension 3.*

A separable metric space X is said to be *locally triangulable at x* if there exists a complex K and a homeomorphism h of $|K|$ into X such that x lies in the interior of $h(|K|)$. A locally triangulable space is one which is locally triangulable at each of its points. The general triangulation problem reads as follows: "Can every locally triangulable space be triangulated?" The author solves the problem for spaces of dimension not greater than 3. The method used consists essentially of reducing the problem to the triangulation problem for 3-manifolds with boundary, or rather, to the more general result proved in Theorem 8' of the paper, *Locally tame sets are tame*, by R. H. Bing [Ann. of Math. vol. 59 (1954) pp. 145–158]. (Received September 29, 1955.)

73. L. M. Sonneborn: *A coincidence theorem for functions on product spaces.*

Let X and Y be compact, arc-wise connected topological spaces. Let $f: X \rightarrow R$, $g: Y \rightarrow R$, and $h: X \times Y \rightarrow R$ be real-valued, continuous functions such that the range of h is contained in both the range of f and the range of g . Then there exist $x \in X$ and $y \in Y$ such that $f(x) = g(y) = h(x, y)$. (Received October 5, 1955.)

74. H. C. Wang: *Fundamental groups of solvmanifolds.*

By a solvmanifold is meant a coset space of a connected solvable Lie group. Mostow has proved that the topological type of a solvmanifold M is determined by its fundamental group $\pi_1(M)$. Moreover if M is compact, then $\pi_1(M)$ determines M up to a homeomorphism. It is the aim of this paper to characterize the class of groups $\pi_1(M)$ as abstract groups. Let N be a finitely generated, nilpotent group without torsion, and H a finitely generated, free abelian group. Any extension of N by H is called a special \mathfrak{S} -group. It is shown that (i) Any special \mathfrak{S} -group is the fundamental group of a solvmanifold and conversely; (ii) Any special \mathfrak{S} -group has a normal subgroup T of finite index such that T can be imbedded, as a uniform discrete subgroup, in a connected and simply-connected solvable Lie group.

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