THE APRIL MEETING IN CHICAGO

The four hundred eightieth meeting of the American Mathematical Society was held at the University of Chicago on Friday and Saturday, April 25–26, 1952. There were about 270 registrations, including the following 229 members of the Society:

W. R. Allen, J. W. Armstrong, K. J. Arnold, Louis Auslander, W. L. Ayres, Reinhold Baer, W. R. Ballard, P. T. Bateman, A. F. Bausch, Henry Beiman, Leon Benson, Gerald Berman, S. F. Bibb, R. H. Bing, R. P. Boas, W. M. Boothby, Raoul Bott, D. G. Bourgin, Joseph Bram, Richard Brauer, R. H. Bruck, R. C. Buck, P. B. Burcham, A. S. Cahn, A. P. Calderón, R. H. Cameron, K. H. Carlson, W. B. Caton, Lamberto Cesari, K. T. Chen, E. W. Chittenden, H. M. Clark, F. M. Clarke, M. D. Clement, Harvey Cohn, E. G. H. Comfort, A. H. Copeland, Sr., V. F. Cowling, E. H. Crisler, C. W. Curtis, M. M. Day, B. V. Dean, John DeCicco, J. C. E. Dekker, W. E. Deskins, Allen Devinatz, Flora Dinkines, W. F. Donoghue, L. A. Dragonette, Roy Dubisch, W. F. Eberlein, B. J. Eisenstadt, H. M. Elliott, Benjamin Epstein, M. H. M. Esser, H. P. Evans, R. L. Evans, Trevor Evans, H. S. Everett, G. M. Ewing, E. R. Fadell, Chester Feldman, R. C. Fisher, Isidore Fleischer, J. S. Frame, Evelyn Frank, L. E. Fuller, R. E. Fullerton, M. P. Gaffney, David Gilbarg, A. M. Gleason, Casper Goffman, H. E. Goheen, Michael Golomb, A. W. Goodman, S. H. Gould, L. M. Graves, R. L. Graves, E. L. Griffin, T. E. Hagensee, Franklin Haimo, P. C. Hammer, Gerald Harrison, H. L. Harter, Charles Hatfield, L. J. Heider, R. G. Helsel, I. N. Herstein, J. J. L. Hinrichsen, I. I. Hirschman, Vaclav Hlavatý, D. L. Holl, T. C. Holyoke, S. P. Hughart, Ralph Hull, J. R. Isbell, W. E. Jenner, Meyer Jerison, R. E. Johnson, G. K. Kalisch, Samuel Kaplan, Irving Kaplansky, William Karush, Chosaburo Kato, M. W. Keller, J. B. Kelly, L. M. Kelly, J. H. B. Kemperman, D. E. Kibbey, Fred Kiokemeister, S. C. Kleene, Erwin Kleinfeld, Fulton Koehler, L. A. Kokoris, Marc Krasner, A. H. Kruse, M. Z. Krzywoblocki, E. P. Lane, R. E. Langer, Leo Lapidus, C. G. Latimer, J. R. Lee, R. A. Leibler, R. B. Leipnik, D. J. Lewis, S. D. Liao, B. W. Lindgren, O. I. Litoff, T. C. Littlejohn, A. J. Lohwater, Lee Lorch, M. F. McFarland, J. D. McKnight, C. C. MacDuffee, Saunders MacLane, Morris Marden, E. P. Merkes, J. M. Miller, J. M. Mitchell, M. A. Moore, G. W. Morgenthaler, E. J. Moulton, H. T. Muhly, S. B. Myers, W. M. Myers, Jr., Zeev Nehari, O. M. Nikodým, Katsumi Nomizu, E. A. Nordhaus, E. P. Northrop, R. J. Nunke, E. J. Olson, E. H. Ostrow, R. R. Otter, M. H. Payne, Sam Perlis, J. K. Peterson, L. E. Pursell, A. L. Putnam, C. R. Putnam, Gustave Rabson, R. A. Raimi, O. W. Rechard, P. V. Reichelderfer, Haim Reingold, R. B. Reisel, Daniel Resch, P. R. Rider, G. S. Ritchie, R. A. Roberts, Alex Rosenberg, P. C. Rosenbloom, Arthur Rosenthal, W. C. Royster, H. J. Ryser, R. G. Sanger, A. C. Schaeffer, H. M. Schaerf, O. F. G. Schilling, Lowell Schoenfeld, J. R. Schoenfield, W. T. Scott, D. H. Shaftman, M. E. Shanks, V. L. Shapiro, S. S. Shu, Edward Silverman, R. J. Silverman, M. F. Smiley, A. H. Smith, E. S. Sokolnikoff, E. H. Spanier, E. J. Specht, George Springer, H. E. Stelson, B. M. Stewart, J. A. Sullivan, C. T. Taam, H. P. Thielman, R. M. Thrall, W. J. Thron, E. F. Trombley, C. K. Tsao, S. M. Ulam, W. R. Utz, N. H. Vaughan, Thirukkannapuram Vijayaraghavan, Bernard Vinograde, G. L. Walker, D. W. Wall, Sylvan Wallach, M. J. Walsh, S. E. Warschawski, D. R. Waterman, M. S. Webster, M. T. Wechsler, André Weil, L. M. Weiner, H. H. Wicke, L. R. Wilcox, K. G. Wolfson, W. D. Wood, F. B. Wright, L. C. Young, P. M. Young, J. W. T. Youngs, Daniel Zelinsky, J. L. Zemmer, Antoni Zygmund.

By invitation of the Committee to Select Hour Speakers for Western Sectional Meetings, Dr. S. M. Ulam of the Los Alamos Scientific Laboratories addressed the Society at 2 P.M. Friday on *Combinatorics and analysis.*

There were a total of eight sessions for the presentation of contributed papers. Three were held on Friday at 10:30 A.M., two on Friday at 3:15 P.M., and three on Saturday at 10:30 A.M. One of the concurrent sessions on Saturday morning was a special session for the presentation of papers which arrived in the Providence office after the deadline. Presiding officers at the various sessions were Professors Richard Brauer, R. H. Bruck, R. H. Cameron, Michael Golomb, L. M. Graves, S. C. Kleene, Zeev Nehari, E. H. Spanier, and J. W. T. Youngs.

Abstracts of the papers read follow. Those having the letter "*t*" after their numbers were read by title. Paper number 403 was read by Professor Copeland, number 411 by Dr. Rosenberg, number 430 by Dr. Hammer, number 441 by Professor Warschawski, number 451 by Professor Ewing, and number 452 by Mr. Cahn. Mr. Blair was introduced by Professor M. F. Smiley, Mr. Feit by Professor R. M. Thrall, Mr. Fettis by Professor P. R. Rider, and Mr. Allen by Professor O. H. Hamilton.

ALGEBRA AND THEORY OF NUMBERS

400. R. L. Blair: *Ideal lattices and the structure of rings.*

A ring A is said to satisfy condition C or D (C_r or D_r) in case the lattice of ideals (right ideals) of A is complemented or distributive, respectively. The principal results are the following. A ring A which is semi-simple in Jacobson's sense (Amer. J. Math. vol. 67 (1945) pp. 300–320) satisfies C (C_r) if and only if A is isomorphic with the discrete direct sum of simple rings (of simple rings with minimal right ideals). If A satisfies C_r, then $A = M \dotplus M^*$, where M is the maximal regular ideal of A (Brown and McCoy, Proceedings of the American Mathematical Society vol. 1 (1950) pp. 165–171) and M^* is the annihilator of M. The rings M and M^* satisfy C_r and M^* is bound to its radical (M. Hall, Trans. Amer. Math. Soc. vol. 48 (1940) pp. 391–404) which is nilpotent of index two. An ideal (right ideal) I of A is *s-irreducible* in case $B \cap C \subseteq I$ for ideals (right ideals) B, C of A implies that $B \subseteq I$ or $C \subseteq I$. (Cf. L. Fuchs, Comment. Math. Helv. vol. 23 (1949) pp. 334–341.) A ring A satisfies D (D_r) if and only if each ideal (right ideal) of A is the intersection of all s-irreducible ideals containing it. Finally, if A is a semi-simple ring which satisfies D_r, then A is isomorphic with a subdirect sum of division rings. (Received March 12, 1952.)

401. K. T. Chen: *A group ring method for finitely generated groups.*

Let G be a finitely generated group, and J a ring with unit element. One may assume $G = F/R$ where R is a normal subgroup of a free group F with free generators $x_1, \cdots, x_m$. Let a factor group M/N, $F \supset M \supset N$, be an invariant of G such that, if G has another presentation, the correspondingly defined factor group is isomorphic with M/N. For example, $M/N=[F, F]/[F, R]$ is such a factor group. Then the quotient ring $JM/(N-1)_M$ is also an invariant of G, where $(N-1)_M$ is the ideal generated by all $u-1$, $u\in N$, in JM. Let $X_1, \cdots, X_m$ be noncommutative indeterminates. Denote by Λ the ring of all formal power series in the form $\sum_{p=0}^{\infty} a_p$, $a_p=\sum \mu_{i_1}\ldots_{i_p} X_{i_1}\cdots X_{i_p}$, $\mu_{i_1}\ldots_{i_p}\in J$. Define ϕ: $JF\to\Lambda$ such that $\phi(x_i)=1+X_i$ and $\phi(\mu a)=\mu\phi(a)$, $\mu\in J$. Then $\phi(JM)/\phi((N-1)_M)$ is an invariant of G. (Received March 13, 1952.)

402. F. Marion Clarke: *Note on quasi-regularity and the Perlis-Jacobson radical.*

The Perlis-Jacobson use of quasi-regularity to characterize the radical of an "arbitrary" algebra or ring R [Bull. Amer. Math. Soc. vol. 48 (1942) pp. 128–132; Amer. J. Math. vol. 67 (1945) pp. 300–320] is shown to require any one of the following properties of weak associativity or commutativity: P1. If x and z are respectively left and right quasi-inverses of a quasi-regular element y of a right quasi-regular right ideal of R, then $(xy)z=x(yz)$; P2. The left and right quasi-inverses of a quasi-regular element of a right quasi-regular right ideal of R are equal and unique; P3. If R has odd characteristic and $R^{(+)}$ consists of the same elements with the same addition as in R but with multiplication defined $r\cdot s=(rs+sr)/2$, then a quasi-regular element of $R^{(+)}$ has the same unique right quasi-inverse in R and in $R^{(+)}$. It is shown that P1, P2, and P3 are equivalent in rings of odd characteristic and that analogous properties are equivalent with reference to any quasi-regular element of R. (Received March 12, 1952.)

403. A. H. Copeland and Frank Harary: *A characterization of implicative Boolean rings.*

An implicative Boolean ring is defined in terms of a cross-product operation and the usual Boolean operations. However it is shown in this paper that such rings can be characterized in terms of familiar ring concepts only. More specifically, a Boolean ring B can contain such a cross-product if and only if it is isomorphic to its quotient rings modulo the non-unit principal ideals. The isomorphisms enable one to set up a semigroup of transformations (not necessarily unique) of B into B. These are one-parameter transformations where the parameter is a nonzero element of the Boolean ring. The product of the transformations defines the cross-product of the parameters. The inverse of one of these transformations, when defined, is an implication which is neither strict nor material. The implication can be extended to elements for which the inverse does not exist and can be given a logical interpretation. (Received March 7, 1952.)

404. W. Feit: *The formula for the degree of the skew representations of the symmetric group.*

By A. Young's well known method, each irreducible representation of the symmetric group on n letters can be associated with a regular diagram containing n nodes, which can be denoted by $(a_1, \cdots, a_m)$ where $\sum a_i=n$, $a_1\geqq\cdots\geqq a_m\geqq 0$; the degree of the associated representation is the number of standard orderings of $(a_1, \cdots, a_m)$. In giving a proof of the Murnaghan-Nakayama recursion formula, G. de B. Robinson considers what he calls skew diagrams and associates with each of these a representation of the symmetric group; such a representation he calls a skew representation. Each skew diagram may be denoted by $(a_1, \cdots, a_m)-(b_1, \cdots, b_m)$ where $a_i \geqq b_i$, $i=1, \cdots, m$, $(a_1, \cdots, a_m)$, $(b_1, \cdots, b_m)$ are regular diagrams. The object of this note is to give the explicit formula for the number of standard orderings of such a skew diagram, which is also the degree of the representation associated with it. The number of orderings of $(a_1, \cdots, a_m)-(b_1, \cdots, b_m)$ is $n! \det (z_{ij})$, where $z_{ij}=1/(a_j-b_i-j+i)!$, $n=\sum a_i-\sum b_i$. The proof of the formula is by induction. (Received March 12, 1952.)

405. J. S. Frame: *Character values for a fixed class in the symmetric groups*. Preliminary report.

To each irreducible representation F_α of the symmetric group S_m corresponds a partition diagram of m nodes arranged in α_1' rows of α_i nodes $(0 \leqq \alpha_{i+1} \leqq \alpha_i)$ and in α_1 columns of α_j' nodes $(0 \leqq \alpha_{j+1}' \leqq \alpha_j')$, where $\sum \alpha_i = \sum \alpha_j' = m$. Define at each node the hook length $h_{ij}=\alpha_i-i+\alpha_j'-j+1$ (that is, one more than the sum of the nodes to the right and below) and the diagonal deviation $d_{ij}=j-i$. Let $h=\prod h_{ij}$ be the product of the m hook lengths and let $s_k=\sum (j-i)^k$ be the sum of the kth powers of the m diagonal deviations d_{ij}. Then the degree of F_α is $f_\alpha = m!/h$. The second main result of the paper is to express as a polynomial in the s_k the common value $\omega_{abc\cdots}$ of the f_α equal characteristic roots of the matrix of F_α that represents the sum of the $g_{abc\cdots}$ elements in that class of S_m whose permutation cycles contain respectively $a+1$, $b+1$, $c+1, \cdots, 1, 1$ symbols. Thus $\omega_0=s_0=m$, $\omega_1=s_1$, $\omega_2=s_2-(m^2-m)/2$, $\omega_2+\omega_{11}=(s_1^2-s_2)/2$, $\omega_3=s_3-(2m-3)s_1$, $\omega_{21}=s_2s_1-4s_3-(m^2-13m+16)s_1/2$, $\omega_3+2\omega_{21}+\omega_{111}=(s_1^3-3s_2s_1+2s_3)/6$, etc. These and similar formulas give the values in each particular class for all the characters $\chi_{abc\cdots}=f_\alpha\omega_{abc\cdots}/g_{abc\cdots}$ of all symmetric groups S_m, and make evident the familiar relations between associated characters. (Received March 13, 1952.)

406. L. E. Fuller: *A canonical form for a matrix over a principal ideal ring modulo m.*

The basic canonical form is for m an integral power of a prime. The elements in these residue class rings are either units or divisors of zero. In the canonical form chosen, every diagonal element is a power of the prime of the modulus and is determined in a prescribed order. Every element is then a multiple of the diagonal of its row; every element above the diagonal is a multiple of the next higher power of the diagonal of its row. Also every element is either "reduced" modulo its column diagonal, or is zero. The Hermite canonical form for a field can be shown to be a special case, although in general this form lacks the triangular property. Uniqueness can be proved by an induction on the order in which the diagonal elements are chosen. To extend this result to the general case, the forms for each of the relative prime factors of m are determined. These are then combined by congruence methods using an extension of the Chinese remainder theorem. (Received February 27, 1952.)

407. I. N. Herstein: *A theorem on rings.*

In his paper *A theorem on division rings* (Canadian Journal of Mathematics vol. 3 (1951) pp. 290–292) Kaplansky proved: Let R be a semi-simple ring with center Z, and suppose that for every x in R some power (depending on x) is in Z $(x^{n(x)} \in Z)$. Then R is commutative. One might well ask what conditions on a ring R with center Z having $x^{n(x)} \in Z$ might prevent commutativity. We prove: Let R be a ring with center Z having $x^{n(x)}$ in Z for all $x \in R$. Then if R is not commutative, the commutator ideal of R must be a nil-ideal. As a consequence we have that if R has no nil-ideals and satisfies $x^{n(x)} \in Z$, then it is commutative. These results contain that of Kaplansky cited above, and in a sense are the best possible. (Received February 25, 1952.)

408. R. E. Johnson: *On ordered domains of integrity.*

Let K^* be the nonzero elements of a domain of integrity K. An element a of K^* is called even if there exist n elements $b, c, \cdots, d$ of K^* such that a is a product of the $2n$ elements $b, c, \cdots, d, b, c, \cdots, d$ in some order. Denote by S the additive semigroup generated by the even elements of K^*. The principal theorem of this paper is that K is orderable if and only if $S \subset K^*$. This extends to a domain of integrity a result of Szele's (*On ordered skew fields*, to appear in the Proceedings of the American Mathematical Society) that a division ring D is orderable if and only if the additive and multiplicative semigroup generated by the nonzero squares of elements of D is contained in D^*. (Received March 3, 1952.)

409. Erwin Kleinfeld: *Simple alternative rings.*

Let R be any alternative ring. If a, b, c are in R and the pairs a, c and b, c are anticommutative, then $(c^2, R)(a, b, c) = 0$. Thus if R is a division ring of characteristic $\neq 2$, squares of associators and commutators are in the center of R. This leads to a relatively short and simple proof that R is either associative or a Cayley-Dickson division algebra over its center, a result previously obtained by R. H. Bruck and the author (Proceedings of the American Mathematical Society vol. 2 (1951) pp. 878–890) and by L. A. Skornyakov (Ukrain. Mat. Zurnal vol. 2 (1950) pp. 70–85), the latter restricting himself to characteristic $\neq 3$. Let S be a simple alternative ring of characteristic $\neq 2, 3$. Then with the aid of the above identity it is shown that S is a Cayley algebra if and only if S contains pairwise anticommutative elements a, b, c such that (a, b, c) is not a divisor of zero. S is a Cayley-Dickson division algebra if and only if S is not associative and $(a, b, S) = 0$ whenever $(a, b) = 0$. The author conjectures that all simple alternative rings are either associative or Cayley algebras. (Received January 17, 1952.)

410. D. J. Lewis: *Cubic Diophantine equations.*

The following theorem is establish: If $F(x_1, x_2, \cdots, x_n)$ is a cubic homogeneous polynomial with rational coefficients and if n is sufficiently large, then $F(x) = 0$ possesses a nontrivial, rational, integral solution. The proof consists in establishing the existence of a nonsingular, integral, linear transformation $x_i = \sum_1^n c_{ij} y_j$ such that $F(x) = G(y)$ and $G(y_1, y_2, \cdots, y_s, 0, \cdots, 0) = \sum_1^s a_i y_i^3$, $n \leqq 6s$. L. G. Peck [Amer. J. Math. vol. 71 (1949) pp. 387–402] has shown that such diagonalized polynomials always have nontrivial solutions provided s is sufficiently large. (Received March 7, 1952.)

411. J. E. McLaughlin and Alex Rosenberg: *Zero divisors and commutativity of rings.*

A Zorn ring is an associative ring in which every non-nil left ideal contains a nonzero idempotent. If the left zero divisors in a Zorn ring A form a proper left ideal, that ideal is the radical of A, and A is a division ring modulo its radical. Conversely, if A is a division ring modulo its radical, the set of left zero divisors is either A or its radical. If a Zorn ring A properly contains its center and all the zero divisors are in the center, then A is a division ring or A is a field F modulo its radical. In the latter case the center of A maps onto a field Z modulo the radical, and F is either a purely inseparable extension of Z or has transcendence degree greater than 1 over Z. This generalizes a theorem of Herstein (Proceedings of the American Mathematical Society vol. 1 (1950) pp. 370–371). If the notion of left zero divisor is replaced by left topological zero divisor, similar theorems hold for Banach algebras. (Received March 11, 1952.)

412t. H. W. E. Schwerdtfeger: *Matrices commuting with their own derivative.*

G. Ascoli has shown (Rendiconti Sem. Mat. Torino vol. 9 (1950) pp. 245–250) that a nonderogatory function matrix $X(t)$ which commutes with its own derivative $X'(t)$ has necessarily all its values commutative: $X(s)X(t)=X(t)X(s)$. In the present paper this theorem is proved again for an *analytic* nonderogatory matrix $X=X(t)$ that satisfies the condition $XX'=X'X$. By making use of induction and of the fact that any two matrices commuting with a nonderogatory X are commutative, it is shown that the derivatives of all orders are commutative: $X^{(i)}X^{(j)}=X^{(j)}X^{(i)}$ $(i, j=0, 1, 2, \cdots)$. If moreover the characteristic polynomial of X is separable, then a *constant* matrix T can be found such that TXT^{-1} is diagonal. The supposition that X is nonderogatory may be dropped if X is a 2×2-matrix, or if $X(t)$ is $n\times n$ and quadratic in t. The same is shown for $n=3$ and $X(t)$ cubic in t. (Received March 6, 1952.)

Analysis

413t. H. D. Block and Buchanan Cargal: *Arbitrary mappings.*

This paper consists of generalizations of some of H. Blumberg's results concerning arbitrary functions (*Arbitrary point transformations*, Duke Math. J. vol. 11 (1944) pp. 671–685). In particular those results concerning the existence of (i) a residual set, each point of which is of homogenous inexhaustible functional approach, and (ii) an everywhere dense set, with respect to which the function is continuous, are shown to hold in much more general types of spaces. As might be anticipated, the proofs are considerably simpler in the more general case. (Received March 10, 1952.)

414t. R. P. Boas and R. C. Buck: *Expansion of analytic functions in polynomial series.* II.

To facilitate the study of the representation of analytic functions by means of series of polynomials, the following terminology is introduced. Given a function $\psi(t)=\sum c_n t^n$ regular at zero and with $c_n\neq 0$ for $n=0, 1, \cdots$, a function $f(z)=\sum a_n z^n$ is said to be of ψ type τ if $\limsup |a_n/c_n|^{1/n}=\tau<\infty$. By means of an integral transform, functions $f(z)$ of ψ type τ correspond one-to-one with functions $F(w)$ which are regular in $|w|>\tau$ and vanish at infinity. This enables the authors to extend earlier results on the representation of entire functions to functions of finite ψ type. For appropriate choices of ψ results are obtained for most of the better known basic sets of polynomials, in particular, for the Laguerre, reversed Laguerre, Hermite, and ultraspherical polynomials. (Received March 6, 1952.)

415. R. C. Buck: *Admissible sequences with vanishing differences.*

A sequence $\{a_n\}$ is admissible if there is an entire function $f(z)$ of exponential type, and of type less than π on the imaginary axis such that $f(n)=a_n$ for $n=0, 1, 2, \cdots$. Let $b_n=\Delta^n a_0=(-1)^n\sum_0^n {}_nC_k b_k$. $\{a_n\}$ is admissible if and only if $\sum b_n z^n$ is regular on the interval $-1\leqq x\leqq 0$. The interpolating function $f(z)$ is then (ML)-$\sum_0^\infty {}_zC_n b_n$ where ML denotes Mittag-Leffler summability. Suppose that $\limsup |a_n|^{1/n}\leqq 1$.

Then, if $b_n=0$ for a set of n of density greater than 1/3, $\{a_n\}$ is admissible; if the density is 1/2, the interpolating function $f(z)$ is of zero type. If $b_n \geqq 0$ for all n, $\{a_n\}$ is again admissible, and f of zero type. This is also true if $b_n \geqq \epsilon > 0$ for all n except a subsequence $\{\lambda_k\}$ for which $\lambda_{k+1}-\lambda_k \rightarrow \infty$. (Received March 14, 1952.)

416. A. P. Calderón: *A general ergodic theorem.*

The validity of the individual ergodic theorem for nonabelian groups of transformations is investigated. Let $\mathcal{E}$ be a measure space of finite total measure and $\mathcal{G}$ a locally compact group of measure-preserving transformations of $\mathcal{E}$ satisfying suitable measurability conditions. Let N_t be a family of compact open symmetric neighborhoods of the identity in $\mathcal{G}$ depending on $t>0$ and such that $N_tN_s \subset N_{t+s}$. Then if $|N_{2t}| < k|N_t|$ where $|N_t|$ stands for the left invariant measure of N_t, there exists a subset R of the reals of density 1 such that for every function $F(x)$ integrable in $\mathcal{E}$ the averages $|N_t|^{-1}\int_{N_t} F(gx)dg$ converge almost everywhere in x as $t \rightarrow \infty$ through R. A dominated ergodic theorem in the form of Pitt holds for these averages. As a consequence of the latter a theorem of Dunford-Zygmund's type is proved. (Received March 12, 1952.)

417t. A. P. Calderón: *A note on invariant measures.*

The following theorem is established. Let $\mathcal{E}$ be a measure space of finite total measure and $\mathcal{G}$ a group of one-to-one transformations of $\mathcal{E}$ preserving measurable sets and sets of measure zero. Then if $\mathcal{G}$ is measurable in the sense of von Neumann, a necessary and sufficient condition in order that there exist a measure invariant with respect to the transformations of the group and absolutely continuous with respect to the given measure μ is that for every measurable set A, $\mu(A)>0$ imply that $\inf \mu(gA) >0$, $g \in \mathcal{G}$. (Received March 12, 1952.)

418. R. H. Cameron: *A generalization of the Poisson formula for the solution of the heat flow equation.*

In this paper a solution is given for the differential equation $\partial^2G/\partial\xi^2 - a\partial G/\partial t + \theta(t, \xi)G=0$ subject to the boundary conditions $G(t, \pm\infty)=0$ and $G(0, \xi)=\sigma(\xi)$. Under suitable smoothness, etc., conditions on θ and σ, it is shown that a solution is given by the Wiener integral $G(t, \xi)=\int_C \exp\{\tau^2\int_0^1\theta[t(1-s), 2\tau x(s)+\xi]ds\} \cdot\sigma[2\tau x(1)+\xi]d_wx$, where $\tau=(t/a)^{1/2}$. This reduces to the Poisson integral solution of the heat flow equation when $\theta \equiv 0$. (Received November 13, 1951.)

419. V. F. Cowling: *On the distribution of the values of the partial sums of a Taylor series.*

The object of this paper is to generalize the results of a preceding paper by the author (Proceedings of the American Mathematical Society vol. 2 (1951) pp. 732–738). In the paper referred to it was shown that if the points of affix $a_ka_{k-1}^{-1}z$, $k=1, 2, \cdots, n$ ($a_k\neq 0$, $k=1, 2, \cdots, n$), lay in a certain region $E=E(V)$ of the complex plane for $z\in Z$, then the values of the partial sums $S_k(z)=a_0+a_1z+\cdots+a_kz^k$, $k=1, 2, \cdots, n$, would be complex numbers such that for $z\in Z$ the points corresponding to these numbers would be contained in a predetermined region V. In the present paper we replace the region E by a number of regions and require only that certain subsets of the quantities $a_na_{n-1}^{-1}z$ fall in these regions for $z\in Z$. The following result is a simple application of these methods. Let $P(z)=a_0+a_1z+\cdots+a_kz^k$ where $a_i\neq 0$, $i=0, 1, \cdots, k$. Suppose $0<r_i\leqq 1$, $i=1, 2, \cdots, k$. Then $P(z)$ has all

its zeros interior to the circle $|z| \leqq \text{Max}\ [((r_1+1)/r_2)|a_0/a_1|, ((r_2+1)/r_3)|a_1/a_2|, \cdots, ((r_k+1)/r_1)|a_{k-1}/a_k|]$. (Received December 24, 1951.)

420. Allen Devinatz: *On positive definite functions.* Preliminary report.

Let Q be a convex set in the z-plane which is symmetric with respect to the real axis and contains the origin. Necessary and sufficient conditions are given for a continuous complex-valued function $f(z)$, defined on Q, to be written in the form $f(z) = \int_{a_1}^{b_1}\int_{a_2}^{b_2} \exp(xt_1+iyt_2)dV(t_1, t_2)$, where x and y are respectively the real and imaginary parts of z and $V(t_1, t_2)$ is a bounded monotone increasing function. The numbers b_1 and b_2 may take on the value ∞, and a_1 and a_2 the value $-\infty$. The main condition is that $\sum_{i,j=1}^{n} \alpha_i\bar{\alpha}_j f(z_j+\bar{z}_i) \geqq 0$ for all finite sets of complex numbers $\{\alpha_1, \cdots, \alpha_n\}$ and all sets $\{z_1, \cdots, z_n\}$ which range over a certain convex subset of Q. The methods employed use the techniques of operators in Hilbert space and the theory of reproducing kernels (N. Aronszajn, Trans. Amer. Math. Soc. vol. 68 (1950)). The results obtained generalize previous results of the author (Bull. Amer. Math. Soc. Abstract 57-1-30), contain a result of M. Krein (C. R. (Doklady) Acad. Sci. URSS. N.S. vol. 26 (1940) pp. 17–22) and contain a corrected version of an incorrect theorem stated by S. Livshitz (C.R. (Doklady) Acad. Sci. URSS. N.S. vol. 44 (1944) pp. 3–7, Theorem 3) for which the author has a counter example. (Received March 13, 1952.)

421. W. F. Donoghue: *Causality and quantum mechanics.* Preliminary report.

A theory is the specification of a linear algebra, R, which is determined by the observables, and a set of states on R. It is physically correct to assume that the linear span, R', of the states is weak-star dense in R^*, the algebraic dual of R, and weak-star bounded there. The Krein-Milman theorem then provides a full set of pure states in R^* if not in R'. The theory is causal (may be embedded in a causal theory) if and only if all of the pure states in R' (R^*) are sharp, in von Neumann's terminology dispersionless. It is established that the causality of a theory (or the possibility of extending it to a causal theory) is equivalent to the commutativity of R. The postulates for quantum mechanics used by von Neumann as well as those suggested by Segal fall into the framework considered, and the celebrated theorem of von Neumann follows from less restrictive hypotheses. Since the causality question reduces to a problem concerning the commutativity of R, it follows that those physical theories which attempt a causal interpretation of quantum mechanics differ from the orthodox formulation of that theory in the specification of R: that is, in the theory of measurement. (Received March 10, 1952.)

422. R. L. Evans: *Solution of linear ordinary differential equations containing a parameter.*

Consider the differential equation (1) $y^{(n)} - \sum_{j=1}^{n} \lambda^{pj} p_j(\lambda, x) y^{(n-j)} = 0$ where: (i) $p_j(\lambda, x) = \sum_{\nu=[(p-\mu-1)/p]}^{\infty} \sum_{\mu=-pm_j}^{\infty} a_{j,\mu,\nu}\lambda^{-\mu}x^{\nu}$ are such that all $p_j(\lambda, x)/\lambda pm_j$ converge wherever $|x| < x_0$ and $|\lambda| > \lambda_0$, (ii) n and p are positive integers and the m_j's are non-negative integers, (iii) $[-\mu/p]$ is the greatest non-negative integer contained in $(-\mu/p)$, and (iv) x is a complex variable and λ is a large complex parameter. Equation (1) has a regular point at $x=0$ and it may also be said to have a turning point there if one or more m_j exceeds zero. If the given initial conditions define (2) $y^{(j)}(\lambda, 0) = (j!) \sum_{\alpha=0}^{\infty} b_{\alpha,j}\lambda^{-\alpha}$ $(j=0, 1, \cdots, n-1)$ convergent whenever $|\lambda| > \lambda_0$, then (1) and (2) together have a solution of the form (3) $y(\lambda, x) = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} b_{\alpha,\beta}\lambda^{p\beta-\alpha}x^{\beta}$, which converges wherever $|x|<x_0$ and $|\lambda|>\lambda_0$. The formula for computing the unknown $b_{\alpha,\beta}$'s is given. This new application of old results yields a solution of a broad class of initial value problems, including many involving turning points. Examples of such problems are presented. (Received April 10, 1952.)

423*t*. Seymour Ginsburg: *On a relation between sets and elements.*

Let E be any nonempty set and $H=\{h\}$ a family of subsets of E. Let R be a relation such that xRH holds for one and only one element x in each h, and G_α the subset of E consisting of those elements x in E for which xRH holds for at least $\aleph_\alpha$ sets h in H. Denote by $p(A)$ the power of A. If H is the family of all finite subsets of the uncountable set E and $\aleph_\alpha<p(E)$, then $p(G_\alpha)=p(E)$. If $\aleph_\alpha=p(E)$, then G_α may be empty. This modifies a result of Fodor and Ketskemety (Fund. Math. vol. 37). If H is the family of those subsets of E which are of power $\aleph_\gamma$ each, where $\aleph_\gamma\leqq p(E)$, and if $\aleph_\alpha=p(E)$, then $p(G_\alpha)=p(E)$. (Received March 3, 1952.)

424*t*. Seymour Ginsburg: *On the number of distinct sums of transfinite series obtained by permuting the elements of a transfinite sequence.*

Let $N(\sum_{\xi<\lambda}a_\xi)$ be the number of distinct sums of all the λ-type series obtained by permuting the elements of the sequence of ordinal numbers $\{a_\xi\}_{\xi<\lambda}$. Generalizing two results of Sierpinski (Fund. Math. vol. 36) it is shown that if ω_ρ is a regular ordinal, then $N(\sum_{\xi<\omega_\rho}a_\xi)\leqq\sum_{\xi<\rho}\aleph_\rho{}^{\aleph_\xi}$. Furthermore if $\omega_\rho=\omega_{\alpha+1}$, then there exists an ω_ρ sequence $\{a_\xi\}$ such that $N(\sum_{\xi<\omega_\rho}a_\xi)=\aleph_\rho$. If ω_ρ is a regular ordinal and $\{a_\xi\}_{\xi<\omega_\rho}$ is a nondecreasing sequence of ordinals, then $N(\sum_{\xi<\omega_\rho}a_\xi)=1$. (Received March 3, 1952.)

425*t*. Seymour Ginsburg: *Ordinal inequalities.*

Let $\{\alpha_\xi\}_{\xi<\lambda}$ and $\{\beta_\xi\}_{\xi<\lambda}$ be two increasing sequences of ordinal numbers whose limits are α and β respectively. In terms of the Cantor normal form of α and of β, necessary and sufficient conditions are given for the limit of the sum and the product of the two sequences to equal the sum and the product of the limits of the sequences. (Received March 3, 1952.)

426*t*. Seymour Ginsburg: *Real functions on posets.*

Let P be a poset with no minimal element. A single-valued, real function on P has a limit a if, to each element p_0 of P, and $\epsilon>0$, there corresponds an element $p_1(p_0,\epsilon)$ of P such that $p_1\leqq p_0$, and $|f(p)-a|<\epsilon$ for $p\leqq p_1$. The usual results on sum, difference, multiplication, and quotient of limits hold. In addition, a poset P can be imbedded isomorphically into an everywhere branching poset Q (Day, *Oriented systems*, Duke Math. J. vol. 11) by an isomorphism g. Furthermore, a function h of Q onto $g(P)$ can be found which has the following two properties: (1) $h[g(p)]=g(p)$; and (2) if f is any function on P, then the function f_*, which is defined by $f_*[g(p)]=f(p)$ for p in P, and $f_*(q)=f_*[h(q)]$ for q in Q, has a limit a on Q if and only if f has a limit a on P. Thus, in a certain sense the real functions on everywhere branching posets are the most general as regards limits. (Received March 10, 1952.)

427. R. L. Graves: *The Fredholm theory in Banach spaces.*

It is possible to generalize the qualitative, nondeterminental aspects of the classical Fredholm theory to a completely continuous operator, K, on any Banach space; in particular the resolvent may be expressed as the quotient of two entire functions, say $R(z)=\text{DET }(1-zK)/\det(1-zK)$. It has been pointed out that the coefficients of the particular entire functions used by Fredholm in the classical case may be expressed in terms of the traces of the interates of the operator, where trace $K = \int_0^1 K(x, x)dx$. For an operator, K, on a general space, define trace K as the sum of the characteristic values (with multiplicity) and trace $|K|$ as the sum of the corresponding absolute values. When trace $|K| < \infty$, this definition of the trace makes the Fredholm formulas applicable, and only minor modifications are necessary when trace $|K^p| < \infty$. Further, if $\|K_q - K\| \to 0$, where K_q is finite ranged, and trace $|K_q| \leq M$, then $\det_2 (1 - zK_q) \to \det_2 (1 - zK)$ and $\mathrm{DET}_2 (1 - zK_q) \to \mathrm{DET}_2 (1 - zK)$ uniformly on compact sets. (These entire functions correspond to those used by Carleman.) Again, minor modifications give similar results when trace $|K_q^p| \leq M$. This yields as particular consequences the determinental results of Fredholm and Carleman. (Received March 11, 1952.)

428. E. L. Griffin: *On semi-finite rings*.

Let M be a ring of operators on a Hilbert space H of arbitrary dimension, with M' as commutant. If M and M' are finite rings, then there exists a positive definite operator C belonging to the center of M which generalizes von Neumann's invariant C for factors. Although the trace in M is always strongest continuous, it is strongly (weakly) continuous if and only if C is a bounded operator. If two such rings are *-isomorphic in such a way that their coupling operators correspond under the isomorphism, then there exists a linear isometry implementing the isomorphism. Further, one can prove that: 1. The strongest topology is purely algebraic. 2. The strong and weak topologies are not purely algebraic but depend on the coupling operator C. 3. C is the identity if and only if M and M' are conjugate isomorphic. 4. If C is bounded, then each positive linear functional on M is a finite sum of functionals of the form (Ax, x), $A \in M$, $x \in H$. These results can be extended to semi-finite rings (which are rings such that every nonzero projection contains a finite nonzero projection) by introducing ratios of cardinals as formal scalars. (Received March 17, 1952.)

429t. E. L. Griffin: *Isomorphisms of rings of type* III. Preliminary report.

A ring of operators is of type III if each of its nonzero projections is an infinite projection. This paper extends the results of our previous paper *On semi-finite rings* (see preceding abstract) to rings of type III. In particular, the notion of coupling operator is generalized and one obtains the Theorem: *If an isomorphism between two rings of type* III *links the coupling invariants, then the isomorphism is implemented by a linear isometry.* This result leads to the following results: 1. *The strong and weak topologies are purely algebraic in rings of type* III, 2. *The strongest topology is purely algebraic in general rings, and* 3. *The strong and strongest topologies coincide in rings of type* III. The methods used here are essentially the same as those used in our previous paper *On semi-finite rings.* (Received April 8, 1952.)

430. P. C. Hammer and Andrew Sobczyk: *Monotonic set functions and convex sets.*

Let G be a real-valued nondecreasing set function (including possibly $+\infty$ and $-\infty$ in its range) defined for a class $\mathfrak{S}$ of subsets of a set M. To each point x in M let there be associated a class $\mathfrak{S}(x)$ of sets in the domain $\mathfrak{S}$ of G. The association $(x, \mathfrak{S}(x))$ is called *exclusive* if no set in $\mathfrak{S}(x)$ contains x. The association $(x, \mathfrak{S}(x))$ is called *farflung* if y in the complement of $x \in \mathfrak{S}(x)$ implies always that there exists a set $Y \in \mathfrak{S}(y)$ such that $Y \supset X$. A functional $g(x)$ is defined as the supremum of $G(X)$

for all $X \in \mathfrak{S}(x)$. Let $C(r)$ be the set of points of M such that $g(x) \leqq r$. For a farflung and exclusive association $(x, \mathfrak{S}(x))$, $C(r)$ is either the entire space M, or $C(r)$ is the intersection of all sets $Y \in \mathfrak{S}(y)$ for all y such that $G(Y) > r$. In particular, if M is a linear topological space and each element of $\mathfrak{S}(x)$ is closed and convex for every x, then $C(r)$ is a closed convex set. Appropriate properties of continuity and of regularity at infinity are defined for set functions to yield boundedness, closure, and other desired properties of the associated families of convex sets. (Received March 12, 1952.)

431t. P. C. Hammer and Andrew Sobczyk: *Subconvex and quasi-convex functionals.*

A real-valued functional $f(x)$ defined for all points x in a linear space M is said to be subconvex if and only if $f(x_1) = f(x_2)$ implies always that $f(ax_1 + (1-a)x_2) \leqq f(x_1)$, $0 < a < 1$. A functional $f(x)$ is said to be *quasi-convex* if and only if $f(ax_1 + (1-a)x_2) \leqq \max [f(x_1), f(x_2)]$, $0 < a < 1$. The following results are obtained. (1) Every convex functional is subconvex and quasi-convex. (2) Every quasi-convex functional is subconvex. (3) A monotonic nondecreasing function of a quasi-convex functional is quasi-convex functional. (4) Every continuous subconvex functional is quasi-convex. (5) The set of points x such that $f(x) \leqq r$ is a convex set, called the *r-truncated* set of $f(x)$, if $f(x)$ is quasi-convex. (6) Given any one-parameter family of convex sets $C(r)$ subject to certain hypotheses, there exists a quasi-convex function $f(x)$ such that each $C(r)$ is the r-truncated set of $f(x)$. Generalized forms of convexity are defined, and properties of corresponding functionals derived. N.B. Professor W. Fenchel has written that he had also defined *quasi-convexity* in a manuscript yet to be published. (Received March 12, 1952.)

432. I. I. Hirschman: *Some density theorems obtained from the theory of quasi-analytic functions.*

Let $f(x) \in L_p(-\infty, \infty)$ $(1 \leqq p < \infty)$ and let X be a subset of the real line R. We define $T_p(f, X)$ as the closure in $L_p(-\infty, \infty)$ of the finite linear combinations of the functions $f(x-\xi)$, $\xi \in X$. A sufficient condition is given for $T_p(f, X) = T_p(f, R)$. For example if $\|f^{(n)}(x)\|_p \leqq n!(\log (n+e))^n$, $n = 0, 1, \cdots$, and if $\limsup_{t \to \infty} (2/\pi t) \log \log X(t) > 1$ where $X(t)$ is the number of elements of X in $[-t, t]$, then $T_p(f, X) = T_p(f, R)$. Let $M_p(\phi, X)$ be the closure in $L_p(-\infty, \infty)$ of the finite linear combinations of the functions $\phi(x) \exp(ix\xi)$, $\xi \in X$. A sufficient condition is obtained for $M_p(\phi, X) = M_p(\phi, R)$. As an example, if $\phi(x) \exp (|x|/\log (|x|+e)) \in L_p(-\infty, \infty)$ and if $\limsup_{t \to \infty} (2/\pi t) \log \log X(t) > 1$, then $M_p(\phi, X) = M_p(\phi, R)$. These results depend upon the distributions of the zeros of quasi-analytic functions. (Received January 7, 1952.)

433t. Ernest Ikenberry and W. A. Rutledge: *Convergence of expansions in the Hermite polynomials $H_n(hw)$.*

Hille has shown that an expansion of the form $f(z) = \sum_{n=0}^{\infty} f_n H_n(z)$, $z = x + iy$, converges in a well defined strip $-\tau < y < \tau$ in the z-plane (Duke Math. J. vol. 5 (1939) pp. 875–936). In this paper, expansions of the form $g(w) = \exp(-kh^2w^2) \times \sum_{n=0}^{\infty} g_n(h, k) H_n(hw)$, where h and k may be complex numbers, are considered. The region of convergence in the w-plane is a strip of width $2\tau/|h|$, where $\tau = \tau(h, k) = -\limsup \{(1/(2n)^{1/2}) \log (2^n \Gamma[n/2] |g_n(h, k)|)\}$. The equation $\tau(h, k) = 0$, for a given k, defines a curve in the h-plane which divides this plane into two regions: C, the region in which $\tau > 0$, giving values of h for which there exists a strip of convergence in the w-plane, and D, the region in which $\tau < 0$, giving values of h for which there exists no strip of convergence. For the class of functions $P(w)\exp(-aw^2)$, where $P(w)$ is any entire function of order less than two, or of minimal type if of order two, the region C is the region in the h-plane for which $R(k-1-a/h^2)<-1/2$. An optimum value of h, for given k, is obtained by maximizing a function $K(h)$. (To be published in the Journal of Mathematics and Physics.) (Received March 10, 1952.)

434. J. R. Lee: *Addition theorems in abstract spaces.*

This paper discusses the existence and properties of solutions of addition theorems of the form $f_i(\xi_1+\xi_2)=G_i[f_1(\xi_1), \cdots, f_n(\xi_1); f_1(\xi_2), \cdots, f_n(\xi_2)]$, $i=1, \cdots, n$; where $f_i(\xi)$ are functions having values in an arbitrary Banach algebra with unit element, and G_i are analytic or, in some cases, rational functions. The existence of solutions depends on known scalar solutions $\phi_i/(\zeta)$, holomorphic in some neighborhood of $\zeta=0$ and with values in a domain Δ where $G_i[\alpha_1, \cdots, \alpha_n; \beta_1, \cdots, \beta_n]$ are holomorphic. The case for $n=1$ was discussed by N. Dunford and E. Hille [Bull. Amer. Math. Soc. vol. 53 (1947) pp. 799–805] where, also, the assumption that $\phi(0)$ belongs to the domain of holomorphism of $G(\alpha, \beta)$ was essential. Certain addition formulas that were excluded from consideration previously may now be handled by making use of a set of n formulas. (Received March 11, 1952.)

435. R. B. Leipnik: *Extension of sizes.*

If F is a set of subsets of A, let $\pi F=\bigcap_{B\in F} B$, $\sigma F=\bigcup_{B\in F}B$, $\delta F=\sigma F-\pi F$. Let R be the reals (or a lattice ordered abelian group). A function μ upon subsets of A to R is called a size if and only if $0\leq\mu(\delta F)\leq\sum_{B\in F}\mu(B)$ whenever $\delta F\in$ domain μ and $F\subset$ domain μ. By restricting suitably the domain of μ, the concepts measure, outer measure, dimension, metric, and generalized Menger metric are included. A general extension theorem is proved for sizes, using metric and measure methods. (Received March 14, 1952.)

436*t*. Josephine M. Mitchell: *The kernel function in the geometry of matrices.* II.

Let D be the domain defined by $Z\overline{Z}'<I$, where Z is a rectangular $m\times n$ matrix of complex numbers, $\overline{Z}'$ its conjugate transpose matrix, and I the identity matrix. We already have shown that the kernel function of the domain D is $K(Z, \overline{Z}')=V^{-1}[\det(I-Z\overline{Z}')]^{-m-n}$ $(Z\in D)$, where V is the Euclidean volume of D (Bull. Amer. Math. Soc. Abstract 57-4-343). Here the asymptotic behaviour of $K(Z, \overline{Z}')$ is studied as Z approaches a point Z_0 on the boundary of D. It is found that if $\det(I-Z_{0j}\overline{Z}'_{0j})=0$ $(r+1\leq j\leq m)$ but $\det(I-Z_{0r}\overline{Z}'_{0r})\neq 0$, Z_{0j} being the matrix consisting of the first j rows of Z_0, then $\lim_{Z\to Z_0} d^{m+n}K(Z, \overline{Z}')=V^{-1}[2\det(I-Z_{0r}\overline{Z}'_{0r})]^{-m-n}$, where d is a certain positive definite quadratic form in Z and $Z\to Z_0$ in a specified manner. The case in which Z_0 is a unitary matrix is also studied. Also an asymptotic form is obtained using the characteristic roots of the matrix $Z\overline{Z}'$ in place of $\det(I-Z_{0j}\overline{Z}'_{0j})$. Similar results hold for symmetric and skew symmetric matrices. (Received March 11, 1952.)

437. G. W. Morgenthaler: *A distribution theorem.*

Let $\{\phi_n(x)\}$ be any uniformly bounded system of orthonormal, real-valued functions on a finite interval $[a, b]$. Let $\{a_k\}$ be an arbitrary real sequence satisfying $a_N=o(A_N)$ where $A_N=(a_1^2+a_2^2+\cdots+a_N^2)^{1/2}\to\infty$. Then independently of $\{a_k\}$

there exists a subsequence $\{\phi_{n_k}(x)\}$ and a bounded function $f(x)$ such that for any set $E\subset[a, b]$, $m(E)>0$, the distribution function of $A_N^{-1}\sum_{k=1}^{N} a_k\phi_{n_k}(x)$ on E tends to a limit distribution whose characteristic function is $(m(E))^{-1}\int_E \exp\{-(\lambda^2/2)f(x)\}dx$, as $N\rightarrow\infty$. (Received March 17, 1952.)

438t. G. W. Morgenthaler: *Theorems on series of Walsh functions.*

Let $\{\psi_n(x)\}$ be the Walsh orthonormal function defined on $[0, 1]$, and let $S_n(x)$ and $b_n(x)$ denote respectively the nth partial sum and $(C, 1)$ mean of the series $(S)\sum_0^\infty a_k\psi_k(x)$. By "$\dotplus$" is meant the operation of addition as defined by N. J. Fine, and $\{n_k\}$ shall always refer to a lacunary sequence $n_{k+1}/n_k>q>1$, of positive integers. The following results, analogous to well known results for trigonometric series, can be established: (1) If $S_n(\xi)\rightarrow s$ and $0\leqq\alpha_n=O(1/n)$, then $S_n(\xi\dotplus\alpha_n)\rightarrow s$ provided the sequence $\{(\xi\dotplus\alpha_n)\}$ is free of dyadic rationals from some place on. (2) If (S) is the Fourier series of $f(x)$ in L, and $\bar{s}(x)=[\limsup_n S_n(x)]<\infty$, then $\underline{s}(x)=[\liminf_n S_n(x)]>-\infty$, and $f(x)=2^{-1}\{\bar{s}(x)+\underline{s}(x)\}$ p.p. (3) If $\sum_0^\infty \alpha_k^2<\infty$, then there exists a "continuous" $f(x)$ (i.e. $|f(x\dotplus t)-f(x)|$ is small with $t>0$) corresponding to each sequence $\{n_k\}$ such that at the place n_k the Fourier coefficient of $f(x)$ is α_k. (4) If the partial sum of the series $\sum_{k=0}^\infty a_k\psi_{n_k}(x)$ oscillate finitely at each point of an interval, then $\sum_{k=0}^\infty |a_k|<+\infty$. (5) If $\sum_{k=0}^\infty a_k\psi_{n_k}(x)$ converges (or is only summable by a Toeplitz method) on a set E, $m(E)>0$, then $\sum_0^\infty a_k^2<\infty$. (6) The distribution function of $A_N^{-1}\sum_{k=0}^N a_k\psi_{n_k}(x)$ relative to a fixed set E, $m(E)>0$, converges to the Gaussian distribution with mean value zero and unit dispersion, provided $a_N=o(A_N)$, $A_N=(a_1^2+a_2^2+\cdots+a_N^2)^{1/2}$. (Received March 17, 1952.)

439. W. M. Myers: *A functional associated with a continuous transformation.*

Let T: $z=t(w)$, $w\in R_0$, be a continuous transformation from a simply-connected polygonal region R_0 in the Euclidean plane into Euclidean three-space. Then the transformation T is a representation for a surface S. T. Radó (*Length and area*, Amer. Math. Soc. Colloquium Publications, vol. 30, 1948) defines a functional $a(T)$, which he shows is independent of the representation T for the surface, and he calls $a(T)$ the lower area of the surface. In defining $a(T)$, Radó uses finite collections of disjoint domains. Let $a_1(T), \cdots, a_5(T)$, respectively, denote the functionals which arise when finite collections of (1) disjoint simply-connected polygonal regions, (2) disjoint finitely-connected polygonal regions, (3) simply-connected Jordan regions, with disjoint interiors, (4) finitely-connected Jordan regions, with disjoint interiors, and (5) disjoint simply-connected domains, are used rather than finite collections of disjoint domains in the definition of $a(T)$. It is shown that $a_j(T)=a(T), j=1, \cdots, 5$, and, consequently, any of the above types of collections of sets may be used in place of finite collections of disjoint domains in defining the lower area $a(T)$. (Received March 10, 1952.)

440. P. C. Rosenbloom: *Perturbation theory for linear operators in a Banach space.*

Let T_0 be a linear transformation of the Banach space X into itself, let x_0 be an eigen-vector corresponding to the eigen-value λ_0, and let x_0^* be an eigen-vector of the adjoint transformation T_0^* with the same eigen-value. Let X_1 be the set of x such that $x_0^*(x)=0$, and suppose that $T_0-\lambda_0 I$, as a transformation of X_1 into itself, has a bounded inverse. It is shown that if U is a bounded linear transformation of X into itself such that $\|U\|$ is sufficiently small, then there exist a unique eigen-vector $x(U)$ and corresponding eigen-value $\lambda(U)$ of the transformation T_0+U which are close to x_0 and λ_0 respectively and such that $x_0^*(x(U))=x_0^*(x_0)$. Effective methods for computing $x(U)$ and $\lambda(U)$ are given and the rate of convergence is estimated explicitly. These functions have Fréchet differentials at $U=0$; the formulas for these contain as special cases those of Born, Heisenberg, and Jordan for Hermitian operators in Hilbert space. Explicit estimates are obtained for the error in using these formulas for approximations to $x(U)-x_0$ and $\lambda(U)-\lambda_0$. These results are related to recent results of L. V. Kantorovich and F. Wolf. (Received March 12, 1952.)

441. P. C. Rosenbloom and S. E. Warschawski: *Polynomial approximation to solutions of variational problems in the complex domain.* Preliminary report.

The mapping function of a simply-connected domain bounded by the curve C can be characterized by the variational problem: to minimize $\int_C |f'(z)|^2 ds$ where f' ranges over the class of functions in $H(2, C)$ such that $f(z_0)=0$, $f'(z_0)=1$. Let $P_n(z)$ be the extremal polynomial when the competing functions are restricted to the polynomials of degree not greater than n. We obtain estimates for the rate of convergence of P_n to f for a very wide class of curves C. This problem is a special case of the best approximation to a given function $f(z)$ in $H(2, C)$ in the sense of L_2 on C by polynomials of degree not greater than n. We obtain an estimate for the rate of convergence in terms of the rate of convergence of the ordinary Fourier expansion of $f[\psi(e^{i\theta})](\psi'(e^{i\theta}))^{1/2}$ on $0\leqq\theta\leqq 2\pi$, where $\pi(w)$ maps $|w|<1$ onto the exterior of C. This result can be extended to the class $H(p, C)$, $p>1$. In these results mild smoothness conditions are imposed on C. Analogous results, however, are also derived for polygonal domains. It should be emphasized that all estimates are given explicitly in terms of the geometrical properties of C. (Received March 12, 1952.)

442. W. C. Royster: *Convexity and starlikeness of analytic functions.*

Necessary and sufficient conditions are found such that a function $f(z)$ regular on an arbitrary arc Γ having a continuously turning tangent maps Γ onto a curve which is convex or starlike with respect to a given point. Some consequences of these conditions are obtained for certain families of curves in the z-plane. (Received February 11, 1952.)

443. V. L. Shapiro: *Square summation and localization of double trigonometric series.*

Let $T_1=\sum_{m,n}a_{mn}e^{i(mx+ny)}$ and $T_2=\sum_{m,n}b_{mn}e^{i(mx+ny)}$ be two double trigonometric series whose coefficients a_{mn} and b_{mn} are $o(1/(|m|+1)^{\lambda}(|n|+1)^{\eta})$, $0\leqq\lambda+\eta\leqq 1$, max $(\lambda, \eta)<1$, $\lambda\eta\geqq 0$. Associate to each series its Riemann function $F_1(x, y)$ and $F_2(x, y)$ obtained by formal double integrations. Thus $F_1(x, y)=a_{00}x^2y^2/4-(y^2/2)\cdot\sum_m'(am_0/m^2)e^{imx}-(x^2/2)\sum_n'(a_{0n}/n^2)e^{iny}+\sum_{m,n}''(a_{mn}/m^2n^2)e^{i(mx+ny)}$. Designate the square partial sums of T_1 by $S_R=\sum_{\max(|m|,|n|)\leqq R}a_{mn}e^{i(mx+ny)}$. The principal result of this paper is that if F_1 and F_2 are equal in any closed domain contained in the interior of the fundamental square Ω, then T_1 is square equisummable $(C,(1-(\lambda+\eta)))$ with T_2 uniformly in any interior closed sub-domain. The method of proof is based on the notion of formal products as developed by Rajchman and Zygmund. This paper also shows that equisummability cannot be carried beyond $(C,1)$ through the use of formal products, and consequently that localization with square summation is not comparable to that using circular summation. (Received March 12, 1952.)

444t. V. L. Shapiro: *Summability of double trigonometric integrals.*

Let ϕ be a complex-valued additive function of a set (B) on every figure in the plane. Furthermore let the total variation of ϕ over a circle of radius 1/2 with center at (u, v) be $o((u^2+b^2)^{\gamma/2})$, $\gamma>-1$. The first principal result of this paper is that there exists a trigonometric series $T=\sum_{m,n} a_{mm}e^{i(mx+ny)}$ with coefficients $a_{mn}=o((m^2+n^2)^{\gamma/2})$ such that T is circularly equisummable $(C, \gamma+1)$ with the integral $\int_{R_2}e^{i(xu+yv)}d\phi(u, v)$. The second result concerns itself with square equisummability where the total variation of ϕ over a square of side 1 with center at (u, v) is $o(1/(|u|+1)^{\lambda}(|v|+1)^{\eta})$ and where $a_{mn}=o(1/(|m|+1)^{\lambda}(|n|+1)^{\eta})$, $0\leqq\lambda+\eta\leqq 1$, $\max(\lambda, \eta)<1$, $\lambda\eta\geqq 0$. It is shown that T is square equisummable $(C, 1-(\lambda+\eta))$ with the integral $\int_{R_2}e^{i(xu+yv)}d\phi(u, v)$. The method of proof involves the notions of formal products and of convolutions of set functions and follows the pattern set forth by Zygmund in his paper *On trigonometric integrals* where analogous results for the real line were first obtained. (Received March 12, 1952.)

445t. M. F. Smiley: *Right H^*-algebras.*

Following a suggestion of H. T. Muhly, Ambrose's definition of H^*-algebras (Trans. Amer. Math. Soc. vol. 57 (1945) pp. 364–386) is weakened by requiring that only the right multiplications R_x have adjoints R_{x^*}. The resulting system is called a *right H^*-algebra A*, and A is called *proper* in case A has no nonzero right annihilator and the mapping $x\rightarrow x^*$ is continuous. Examples of proper right H^*-algebras which are not H^*-algebras are easily given by introducing an equivalent norm in a (full) matric H^*-algebra. Conversely, minor modifications of the proof of Ambrose show that every proper right H^*-algebra A is obtained from a proper H^*-algebra A_1 by introducing an equivalent norm in each of the simple components of A_1. Hence proper right H^*-algebras are *dual* in the sense of Kaplansky (Ann. of Math. vol. 49 (1948) pp. 689–701). (Received March 13, 1952.)

446t. C. T. Taam: *On nonoscillatory differential equations.*

Let $f(x)$ belong to $L(0, R)$ for every large R. Equation (1) $y''+f(x)y=0$ is called disconjugate in an interval I if each solution $(\neq 0)$ of (1) has not more than one zero in I. If I is either closed or open, and if open it need not be bounded, it is proved in this paper that (1) is disconjugate in I if and only if there exists some function $\lambda(x)$ which is absolutely continuous on every closed interval contained in I and satisfies $\lambda'+\lambda^2\leqq -f(x)$ almost everywhere in I. Using this criterion the following main result can be easily proved: Assuming that $f(x)$ and $g(x)$ belong to $L(a, \infty)$ and letting (1′) be the equation if $f(x)$ in (1) is replaced by $g(x)$, if (1) is nonoscillatory in I $(a<x<\infty)$ and if $\int_x^\infty f(x)dx\geqq\left|\int_x^\infty g(x)dx\right|$ for $x>a$, then (1′) is also nonoscillatory in I. This is a generalization of a theorem of E. Hille (*Non-oscillation theorems*, Trans. Amer. Math. Soc. vol. 64 (1948) pp. 234–252). An explicit sufficient criterion is given. Also a self-adjoint differential equation of the third order is discussed. (Received March 11, 1952.)

447. C. T. Taam: *The boundedness of the solutions of a differential equation in the complex domain.*

Consider (1) $W''+Q(z)W=0$, where $Q(z)$ is analytic. Let x be real and write $Q(x)=q_1(x)+iq_2(x)$, $q_1(x)$ and $q_2(x)$ being real-valued. Further set $\phi(x)=\int_0^x[|a-q_1(x)|+|q_2(x)|]dx$, where $a>0$. In this paper it is proved that along the positive real axis every solution $W(z)$ $(\neq 0)$ of (1) satisfies $W(x)=O(\exp[2^{-1}a^{-1/2}\phi(x)])$ and $\limsup|W(x)|\exp[2^{-1}a^{-1/2}\phi(x)]>0$ as $x\rightarrow\infty$. Hence every solution $W(x)$ is bounded on the positive real axis if there exists a positive constant a such that $\phi(x)$ is bounded. If $Q(x)$ is real, these reduce to the same results as obtained by N. Levinson (*The growth of the solutions of a differential equation,* Duke Math. J. vol. 8 (1941) pp. 1–10). It is also shown that $W(x)$ behaves like sine function for large x if $\phi(x)$ is bounded. Applying these results to certain regions in the z-plane, sufficient conditions for the boundedness of the solutions $W(z)$ in these regions are then obtained. (Received March 11, 1952.)

448t. D. R. Waterman: *A convergence theorem for Dirichlet series.*

It is known that if $g(z)=\sum c_n z^n$ is analytic in $|z|<1$ and maps a domain in the interior of the circle with an arc ($\alpha\leqq\theta\leqq\beta$) in its boundary into a region of finite area on the Riemann surface of g, then $\sum c_n e^{in\theta}$ is (C, k) summable, $k>0$, almost everywhere in (α, β) if $c_n=o(n^k)$. For $k=0$, this result of Zygmund is seen to be a localization of the celebrated "finite area" principle of Féjer. The condition $c_n=o(1)$ is necessary for the existence of one point of convergence. The same result may be proved for a Dirichlet series $f(s)=\sum a_n e^{-\lambda_n s}$ with the condition on c_n replaced by $\alpha(n)=\max_{0\leqq h\leqq 1}\left|\sum_{\lambda_n\leqq\lambda_k\leqq\lambda_{n+h}} a_n\right|=o(n^k)$. For convergence the condition $\alpha(n)=o(1)$ is shown to be necessary. The principal tool is Zygmund's theory of formal multiplication of integrals which is used to reduce the problem in the half-plane to the problem in the circle. (Received March 13, 1952.)

449. D. R. Waterman: *On an integral of Marcinkiewicz.*

If $\phi(z)$ is of class H_p, $p>0$, one may define $g(\theta)=\{\int_0^1(1-\rho)|\phi'(\rho e^{i\theta})|^2 d\rho\}^{1/2}$, $g^*(\theta)=\{(1/n)\int_0^1(1-\rho)d\rho\int_0^{2\pi}\phi'(\rho e^{it})P(\rho, t-\theta)dt\}^{1/2}$ (P is, of course, the Poisson kernel for the circle), and $s(\theta)=\{\iint_{\Omega_\theta}|\phi'|^2 d\omega\}^{1/2}$ where Ω_θ is the rotation (by angle θ) of a region in the interior of the circle and contained between two chords meeting at $z=1$ to form an angle η. If $f(\theta)$ is of class L_p, $p>1$, one defines $\mu(\theta)=\{\int_0^\pi([F(\theta+t)+F(\theta-t)-2F(\theta)]^2/t^3)dt\}^{1/2}$ where $F(\theta)$ is any primitive of $f(\theta)$. For $g(\theta)$ Littlewood and Paley showed $\|g\|_p\leqq A_p\|\phi\|_p$, $p>0$. For $g^r(\theta)$ they proved an analogous result but for $p\geqq 2$ an integral. This was extended to $p>1$ by Zygmund who also gave this result for the Marcinkiewicz function $\mu(\theta)$ and a result analogous to that on g for $s(\theta)$. Here these results are extended in the same form to functions similarly defined for the Hille-Tamarkin class $\mathcal{H}_p$ in the half-plane. (Received March 13, 1952.)

Applied Mathematics

450t. H. D. Block: *Laws of attraction having a certain generalized Newtonian property.*

Consider a central force law having an associated potential energy of interaction $V=m_1m_2v(r)$ for two point masses of mass m_1 and m_2 a distance r apart, with $v(r)\in C^2$ for $r>0$. The following question is answered. Which of these force laws have the property that two uniform (or more generally, radially symmetric) spheres of radii a and b have as potential energy of interaction $\phi(a, b)m_1m_2v(r)$, where m_1 and m_2 are the masses of the spheres, r is the distance between the centers, and ϕ is a suitable function of the radii. If one specifies in addition that the energy vanish at infinity, or that it be monotone and bounded at infinity, then one finds that only the Coulomb and the Yukawa meson potentials have the required property. The details will appear in the Journal of Mathematics and Physics. (Received March 10, 1952.)

451. H. D. Brunk and G. M. Ewing: *The approximation of double integrals by means of line integrals.*

Given an integral $I=\iint g(r,\theta)drd\theta$, it is shown that a certain line integral I^*_α along a suitable spiral Γ_α of parameter α converges to I as $\alpha\to\infty$. Conditions on the set R over which I is taken, on the integrand g, and on the class of spirals Γ_α which will serve are such as to generalize a result of J. E. Wilkins, Jr. (Bull. Amer. Math. Soc. vol. 55 (1949) pp. 191–192) in all of these directions. In case $g(r,\theta)\equiv p(r)q(\theta)$ and the spiral is $\theta=\alpha\int_0^r p(u)du$, useful bounds on the error $|I-I^*_\alpha|$ are obtained. (Received March 11, 1952.)

452. A. S. Cahn and D. S. Saxon: *Modes of vibration of a suspended chain.*

A method is described for calculating the characteristic frequencies of a suspended inextensible chain vibrating with small amplitude in the plane of the catenary forming the equilibrium configuration. An asymptotic solution of the linearized equations of motion is obtained such that the accuracy of the results increases as the mode number increases and/or as the catenary becomes flatter. (Received February 25, 1952.)

453. H. E. Fettis: *Some properties and applications of the incomplete Bessel function.* Preliminary report.

The function $j_p(z,\theta)$ defined by $j_p(z,\theta)=\int_0^\theta e^{iz\cos\phi}\cos p\phi d\phi$ has recently become important in the theory of unsteady air forces in compressible media at subsonic speeds. The Bessel function of integral order, $J_p(z)$, is found as a special case when $\theta=\pi$: $j_p(z,\pi)=\pi(i)^pJ_p(z)$. Various properties of the function $j_p(z,\theta)$, such as differential relations and recurrence formulae, analogous to those already known for the Bessel function, are derived. Some additional functions related to $j_p(z,\theta)$ are also discussed, and the application of these functions in nonstationary subsonic airfoil theory is indicated. (Received January 8, 1952.)

454. Fulton Koehler: *Estimation of errors in the Rayleigh-Ritz method for eigenvalue problems.*

Let a positive definite, self-adjoint eigenvalue problem be given by the differential equation $Ly=\lambda y$, with boundary conditions. Consider the two variational problems of minimizing the expressions $\int\phi L\phi dP$ and $\int(L\phi)^2dP$ where ϕ is a linear combination of the first n admissible functions of a given set $\psi_1,\psi_2,\cdots$, and $\int\phi^2dP=1$. The roots of the characteristic equations of these problems are denoted by $\mu_1\leqq\mu_2\leqq\cdots\leqq\mu_n$ and $\nu_1^2\leqq\nu_2^2\leqq\cdots\leqq\nu_n^2$; and one has $\lambda_k\leqq\mu_k\leqq\nu_k$, $k=1,2,\cdots,n$, where λ_k is the kth eigenvalue. Let $G(P,Q)$ be the Green's function for the equation $Ly=0$ with the given boundary conditions and let $K(P,Q)=\int G(P,X)G(X,Q)dX$. One then has $\lambda_k^{-2}-\epsilon_n\leqq\nu_k^{-2}\leqq\mu_k^{-2}\leqq\lambda_k^{-2}$, where $\epsilon_n^2=\iint(K-A)^2dPdQ$, and A is the best approximation to K in the L^2 sense by a sum of the form $\sum_{i,j=1}^n a_{ij}L\psi_i(P)L\psi_j(Q)$. If $f_k(P;n)$ and $g_k(P;n)$ are the approximate eigenfunctions corresponding to the roots μ_k and ν_k^2, each will differ in the mean from an eigenfunction corresponding to λ_k by a constant, depending only on k, times ϵ_n. The same is true in the sense of uniform approximation for the function g_k if $\int G^2(P,Q)dQ$ is bounded; and it is true for the function f_k, with ϵ_n replaced by $\epsilon_n^{1/2}$, if $G(P,Q)$ is bounded. The results stated can be extended in slightly modified form to problems of the type $Ly=\lambda\rho y$, where $\rho>0$. (Received March 14, 1952.)

455. M. Z. Krzywoblocki: *On isotropic turbulence in magneto-hydrodynamics of compressible fluids.*

Batchelor and next Chandrasekhar originated and developed the theory of isotropic turbulence in magneto-hydrodynamics of incompressible media. In the present paper the author derives the equations of isotropic turbulence in magneto-hydrodynamics of compressible fluids under the assumption that all the stationary random variables are dependent. (Received February 11, 1952.)

456*t*. M. Z. Krzywoblocki: *On particular integrals of a system of nonlinear partial differential equations.*

A system of nonlinear partial differential equations of arbitrary order and degree by means of a suitable transformation function is transformed into a system of ordinary differential equations. The latter one can be solved by means of computing devices. The proposed method theoretically allows to solve any system of such equations. The only limitations from a practical standpoint are due to the capacity of the existing computing devices. (Received February 11, 1952.)

457. Daniel Resch: *Some Baecklund transformations of second order partial differential equations.*

Consider two surfaces $z=z(x, y)$ and $z'=z'(x', y')$ and four given relations among the first-order elements of these surfaces. If it is possible to find a one-to-one mapping between these surfaces so that the given relations are satisfied at corresponding points, then the relations constitute a Baecklund transformation between the surfaces. If the unprimed quantities and alternatively the prime quantities are eliminated from the given relations, it is possible, in general, to obtain also a correspondence between a partial differential equation in $z(x, y)$ and one in $z'(x', y')$. C. Loewner, *A transformation theory of the partial differential equations of gas dynamics*, NACA TN 2065 (1950) extended the problem to show that systems of partial differential equations can be similarly connected. In the present paper, similar methods are used to find extended Baecklund transformations that will connect second order partial differential equations in three independent variables. If the given relations are linear in the dependent variables z and z', and their first-order derivatives, then they constitute, with certain restrictions on the coefficients of the linear expressions, a Baecklund transformation of a class of hyperbolic partial differential equations into the wave equation in two spacial dimensions. The transformation is in general different from a coordinate transformation. (Received March 11, 1952.)

Geometry

458. E. F. Allen: *A nine-point conic of a triangle.*

A generalization of the nine-point circle configuration is obtained by setting $z=x+ry$, instead of $z=x+iy$ in the inversive geometry of Morley and Morley. It is shown that a nine-point ellipse of a triangle inscribed in an ellipse can be obtained by a general interpretation of the coordinate z as well as by a projective transformation. The line of images of a point on a circle is shown to be a special case of the line of images of a point on a central conic. Perpendicular lines are particular cases of conjugate lines. (Received March 10, 1952.)

459. Gerald Berman: *Finite projective plane geometries and difference sets.*

Let S be the set of residue classes of integers modulo $q=m^2+m+1$, let d_i $(i=0, 1, \cdots, m)$ be a difference set modulo q, and let L_i be the subset of S containing the $m+1$ residue classes corresponding to d_j+i $(j=0, 1, \cdots, m)$. James Singer [*A theorem in finite projective geometry and some applications to number theory*, Trans. Amer. Math. Soc. (1938) pp. 377–385] showed that if the elements of S are taken to be points with collinearity defined by the sets L_i $(i=0, 1, \cdots, q-1)$, then S is a projective plane. In this paper the difference sets corresponding to $PG(2, m)$ where $m=p^n$ (p a prime integer) are studied with the aid of Galois fields, and simple constructions are given for them. The following two conjectures made by Singer are proved: (i) The difference set td_i $(i=0, 1, \cdots, m)$, where $(t, q)=1$, is equivalent to d_i $(i=0, 1, \cdots, m)$ if and only if t is a power of p modulo q. (ii) If d_i $(i=0, 1, \cdots, m)$ and d_i' $(i=0, 1, \cdots, m)$ are any two difference sets modulo q, there exists an integer t for which td_i $(i=0, 1, \cdots, m)$ is equivalent to d_i' $(i=0, 1, \cdots, m)$. This shows that there are exactly $\phi(q)/3n$ inequivalent difference sets corresponding to $PG(2, m)$. (Received March 13, 1952.)

460*t*. D. G. Bourgin: *Separation of convex sets.*

Let E_n be a simplex in R_n. Results are obtained for the separation of pairs of simplexes in R_n by hyperplanes parallel to the faces of E_n. (Received March 13, 1952.)

461*t*. V. G. Grove: *On generalized curvatures.*

Springer and Dekker generalized the geodesic curvature of a curve on a surface. These are called the union curvature, and the hypergeodesic and supergeodesic curvatures. These curvatures are made to depend upon the union curves of a congruence, or hypergeodesic or supergeodesic curves according to the kind of curvatures. We show that these curvatures are but special cases of a more generalized curvature which may be developed with no use whatever of union curves, or hypergeodesic curves or supergeodesic curves. In addition a new curvature, called by us the relative curvature K_λ, is found. In particular if the relative curvature of a curve vanishes at a point, the principal normal of the curve coincides with the line λ with respect to which the relative curvature is defined. Moreover this relative curvature, and in particular the curvatures of Springer and Dekker, may be found from the geodesic curvature of the curve and the normal curvature of the surface in the direction of the tangent to the curve by simple projections. (Received February 18, 1952.)

462*t*. V. G. Grove: *The quadric of Lie.*

A geometrical characterization of a generic point g of the quadric of Lie of a surface S at a point x is found as follows. Let the asymptotic curves and tangents be called the u- and v-curves and u- and v-tangents. As is known, reciprocal lines l_1, l_2 may be defined without the use of the quadrics of Darboux. The line l_2 intersects the u- and v-tangents in points r and s respectively. The line l_1 intersects the loci of r and s as x generates the v-curve and u-curve respectively in points z_1, z_2. Let h be the harmonic conjugate of x with respect to z_1, z_2. Let I be the harmonic conjugate of x with respect to the focal points on l_1. The generic point g is the harmonic conjugate of x with respect to h and I. The locus of g is the quadric of Lie. (Received February 18, 1952.)

463t. P. C. Hammer and Andrew Sobcyzk: *Semispaces in E_n.*

Let x be a point in E_n, $n \geqq 1$, and let $E_n, E_{n-1}, \cdots, E_0 = x$ be a descending set of linear varieties with dimensions given by the subscripts. Let H_k be one of the two k-dimensional relatively open halves of E_k separated by E_{k-1}, for $k = 1, 2, \cdots, n$. The union $S = H_1 \cup H_2 \cup \cdots \cup H_n$ is called a *semispace at x* or merely a *semispace*. Then the following results are obtained. (1) Every semispace is convex. (2) The reflection of a semispace S at x through x is a semispace, called the *complementary semispace* of S. (3) If x is in the complement of a convex set C, then there is a semispace at x which contains C. (4) Every convex set which is not E_n is the intersection set of a class of semispaces. (Received March 12, 1952.)

464t. A. R. Schweitzer: *Theorems in Grassmann's extensive algebra.*

Let $X = x_1E_1 + x_2E_2$ and $Y = y_1E_1 + y_2E_2$ where E_1 and E_2 are perpendicular unit vectors. Then the following theorems are derived from the relation: $\sigma \cdot Q(Y, X) = x_1y_1 + x_2y_2 + (x_1y_2 - x_2y_1)Q(E_2, E_1)$ where $\sigma = x_1^2 + x_2^2$ and $Q(Y, X)$ is an operator on X which transforms X into the vector Y. I. If Y and X have equal lengths, then $Q(Y, X) = \cos(Y, X) + \sin(Y, X) \cdot Q(E_2, E_1)$. That is, $Q(Y, X) = e^{i\theta}$ where $i = Q(E_2, E_1)$ and $\theta = (Y, X)$, the angle measured from X to Y. Also $Q(Y, X)$ is equal to $Q(Y, E_1)$ divided by $Q(X, E_1)$ where $Q(Y, E_1) = y_1 + y_2Q(E_2, E_1)$ and $Q(X, E_1) = x_1 + x_2Q(E_2, E_1)$. II. If $Q(Y, X) = -Q(X, Y)$, then X and Y are perpendicular and have equal lengths. In the preceding it is assumed that the length of a linear vector is positive. (Received March 10, 1952.)

Logic and Foundations

465t. A. R. Schweitzer: *A classification of mathematics.*

The content of mathematics is separated into three categories: I. Pure mathematics, including arithmetic of number systems, algebra, and analysis; II. Applied mathematics, including geometry, topology, Grassmann's extensive algebra, physics, dynamics, and relativity theories due to Minkowski, Einstein, and Eddington; III. Speculative mathematics, including foundations, logic, and philosophy. These categories are not mutually exclusive. Thus Grassmann's algebra may be interpreted (1) as a theory on number systems, (2) as relevant to the foundations of geometry, (3) as including vector calculus, the mathematics of engineering, and tensor calculus, the mathematics of relativity. In the above classification "analysis" includes theories on functional equations such as difference equations, differential equations, integral equations, integro-differential equations, and the author's equations in iterative composition of functions (including "quasi-transitive" equations). A type of functional equations is fundamental for the calculus of variations. (Received March 10, 1952.)

466t. A. R. Schweitzer: *A link between the history of mathematics and the history of philosophy.*

In ancient philosophy number mysticism is prominent. Thus in the Pythagorean school the "tetractys" was considered sacred (compare Burnet, *Early Greek philosophy*, London, 1908, p. 113). In the philosophy of Empedocles reference is made to the four roots of all things (Burnet, loc. cit., p. 264). Also in the Scriptures the number four is frequently mentioned; see, for example, Genesis 2 (10), 13(14); Exodus 28(16), 20(2); II. Samuel 12(6); Job 11(18), (19); Proverbs 30(15), (18); Ezra 1(5), (6); Daniel 7(2), (3); Revelations 4(6), 6(1)–(8), 7(1), 14(7), 21(16); Ephesians 3(18); Luke 19(8). On the mathematical side P. G. Tait (Encyclopedia

Britannica, 11th ed., vol. XXII, p. 718) remarks, "The word 'quaternion' properly means 'a set of four.' In employing such a word to denote a new mathematical method Sir W. R. Hamilton was probably influenced by the recollection of its Greek equivalent, the Pythagorean Tetractys (τετρακὺς, the number four) the mystic source of all things." (Received March 10, 1952.)

TOPOLOGY

467t. D. G. Bourgin: *Topologies for complexes.*

The simplicial complex K is required to be the union of all its n skeletons. Define the full topology of a product space by an open base composed of products of open sets, one in each space. Let K_1 be the barycentric derived of K. If K is imbedded in a product space of closed intervals one for each vertex and the full topology is supposed for the product space, the induced topology in K may be referred to as the convex topology. If K_1 is similarly imbedded and the topology on K is determined by the natural map from the convex K_1 to K, the induced topology may be shown equivalent to the weak topology. The order of fineness is given by geometric, natural, convex, weak. With the aid of the imbedding the key theorem is established that in the weak topology K is paracompact. (Received March 13, 1952.)

468. E. R. Fadell: *Unessential identifications in singular homology theory.*

Let M denote a Mayer complex and M' a subcomplex of M. Radó (T. Radó, *An approach to singular homology theory*, Pacific Journal of Mathematics vol. 1 (1951) pp. 265–290) terms the subcomplex M' an unessential identifier for M provided the natural homomorphism $\pi: M \rightarrow M/M'$ induces an isomorphism onto between the corresponding homology groups. Let R denote the singular complex introduced by Radó (T. Radó, loc. cit.), S the singular complex introduced by Eilenberg (S. Eilenberg and J. A. Zilber, *Semi-simplicial complexes and singular homology theory*, Ann. of Math. vol. 51 (1950) pp. 499–513) and $\sigma: R \rightarrow S$ the natural chain mapping from R to S. Furthermore, let β^R, β^S denote the barycentric homomorphisms of R and S respectively. The main purpose of this note is to show that the nuclei of $\sigma\beta^R$, β^S form unessential identifiers for R and S respectively. The first of these results is an improvement over certain results obtained by Radó and Reichelderfer (T. Radó, loc. cit., P. V. Reichelderfer, *On the barycentric homomorphism in a singular complex.* To appear in the Pacific Journal of Mathematics). (Received January 17, 1952.)

469t. P. C. Hammer and Andrew Sobczyk: *Generalized closure of sets.* Preliminary report.

Let M be a set of points and let $\mathfrak{S}_1$ and $\mathfrak{S}_2$ be classes of subsets of M neither of which includes the null set. Let T be a transformation with domain $\mathfrak{S}_1$ and range $\mathfrak{S}_2$. A subset C of M is called *T-closed* if and only if $X \in \mathfrak{S}_1$, $X \subset C$ implies always that $T(X) \subset C$. The complement of a T-closed set is called a *T-open set.* The following properties follow: (1) The set M and the null set are T-closed and T-open. (2) The intersection set of a class of T-closed sets is T-closed; the union set of a class of T-open sets is T-open. (3) There exists a unique minimal T-closed set containing any subset X of M. This set is called the *T-closure* of X and is designated $\overline{X}$. (4) For every pair of subsets X, Y of M, $\overline{X \cup Y} \supset \overline{X} \cup \overline{Y}$. With proper limitations on M and definitions of $\mathfrak{S}_1$, $\mathfrak{S}_2$, and T, one may obtain ordinary closure, symmetry of any kind, and convexity as special instances. Relations are studied between T-closure and other methods of defining a semi-topology in a set (for examples, suppression of some of the Hausdorff axioms for open sets or neighborhoods, weakening of the requirements for a partial-order topology). (Received March 12, 1952.)

470. S. D. Liao: *On secondary obstruction of sphere bundles.* Preliminary report.

Let $\mathcal{B}$ be an orientable sphere bundle with (i) base space B which is a finite connected complex, (ii) total space E, (iii) fiber S^n, and (iv) projection ϕ. Consider the Gysin's sequence: $\cdots \to^{\rho} H^n(B, Z) \to^{\phi^*} H^n(E, Z) \to^{\sigma} H^0(B, Z) \to^{\rho} H^{n+1}(B, Z) \to^{\phi^*} \cdots \to^{\rho} H^p(B, Z) \to^{\phi^*} H^p(E, Z) \to^{\sigma} H^{p-n}(B, Z) \to^{\rho} \cdots$, where Z = the group of integers. Suppose that the primary obstruction of $\mathcal{B}$ vanishes. Then σ is a homomorphism onto and ϕ^* is an isomorphism into. Let f be a cross section of $\mathcal{B}$ over B^{n+1}(= the $(n+1)$-skeleton of B). Then the secondary obstruction $Z^{n+2}(f)$ is defined. Let $c \in H^n(E, Z)$ be such that $\sigma(c)$ = the unit of the cohomology ring $H(B, Z)$. Then $\theta(f) = c - \phi^* f^*(c) \in H^n(E, Z)$ is defined independently of the choice of c and is a homotopy invariant of f over B^n. The main aim of this paper is to show that (i) $\phi^* Z^{n+2}(f) = \phi^* w^2 \cup \theta(f) - Sq^2 \theta(f)$ for $n > 2$ where w^2 is the 2nd Whitney characteristic class of $\mathcal{B}$; (ii) $\phi^* Z^{n+2}(f) = \theta(f) \cup T^* \theta(f)$ for $n = 2$ where T is the 2-periodic map of E induced naturally by the 2-periodic map of S^2 that sends each point to its antipodal. The corresponding homotopy classification theorems of cross sections over B^{n+1} are also obtained. (Received March 10, 1952.)

471*t*. G. R. Livesay: *Two theorems on the two-sphere.*

(1) Let $S = \{(x, y, z) \in E_3 \mid x^2 + y^2 + z^2 = 1\}$, and let K be a simplicial subdivision of S such that if σ is a simplex of K, then the image of σ under the map p, a reflection in the center of S, is another simplex of K. Let M and N be closed subcomplexes of K such that if $\sigma_M \in M$, $\sigma_N \in N$, then $p\sigma_M \in M$, $p\sigma_N \in N$, and $M + N = K$. If M has no component containing two diametrically opposite points of S, then there are two diametrically opposite points in a component of N. (2) If θ is any angle such that $0 \leqq \theta \leqq \pi/2$, and f is a mapping of S into the reals, then there are two diameters of S, forming an angle θ, at the four end points of which f has the same value. This extends a theorem of F. J. Dyson [Ann. of Math. vol. 54 (1951)]. (Received January 18, 1952.)

472. Katsumi Nomizu: *On the cohomology of compact homogeneous spaces of nilpotent Lie groups.*

It is known by A. Malcev that any compact homogeneous space of a connected nilpotent Lie group can be expressed in the form $M = G/D$ where G is a simply connected nilpotent Lie group and D a discrete subgroup. The first purpose of this paper is to prove that the cohomology algebra of M (with real coefficients) is isomorphic with the cohomology algebra of the Lie algebra g of G. (The isomorphism of the first and second cohomology groups has been shown by Y. Matsushima, Nagoya Math. J. vol. 2 (1951).) From this theorem some results on the Euler characteristic and Betti numbers of M, which are analogous to the known results on homogeneous spaces of compact Lie groups, are obtained. The second purpose is to show that the complex of G-invariant differential forms on M is isomorphic with the complex of invariant cochains of g. From these results an example of a compact homogeneous space of a noncompact Lie group for which the theory of invariant integrals of E. Cartan does not hold is obtained. (Received March 11, 1952.)

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