THE FEBRUARY MEETING IN NEW YORK

The 478th meeting of the American Mathematical Society was held at Columbia University, New York City, on Saturday, February 23, 1952. The meeting was attended by about 200 persons, including the following 175 members of the Society.

Milton Abramowitz, R. D. Anderson, R. L. Anderson, Joseph Andrushkiw, N. C. Ankeny, R. G. Archibald, N. J. Artin, Sholom Arzt, Enrique Bayó, E. G. Begle, D. L. Bernstein, D. W. Blackett, J. H. Blau, Volodymyr Bohun-Chudyniv, Samuel Bourne, J. W. Bower, H. W. Brinkmann, Paul Brock, F. E. Browder, A. B. Brown, L. J. Burton, D. W. Bushaw, J. H. Bushey, Jewell H. Bushey, Eugenio Calabi, W. R. Callahan, Evelyn Carroll-Rusk, P. W. Carruth, G. Y. Cherlin, P. J. Cocuzza, E. A. Coddington, L. W. Cohen, R. M. Cohn, W. H. H. Cowles, M. L. Curtis, R. B. Davis, C. R. DePrima, Jesse Douglas, Beno Eckmann, Samuel Eilenberg, C. C. Elgot, Joanne Elliott, Bernard Epstein, Donald Epstein, M. P. Epstein, Trevor Evans, W. H. Fagerstrom, W. E. Ferguson, Arrigo Finzi, W. T. Fishback, Edward Fleisher, A. D. Fleshler, H. A. Forrester, L. G. Fourès, Gerald Freilich, Bernard Friedman, Murray Gerstenhaber, Abolghassem Ghaffari, B. P. Gill, Sidney Glusman, H. E. Goheen, J. K. Goldhaber, Daniel Gorenstein, J. W. Green, L. W. Green, H. M. Griffin, Emil Grosswald, Laura Guggenbuhl, Felix Haas, G. A. Hedlund, Henry Helson, Aaron Herschfeld, W. M. Hirsch, C. C. Hsiung, S. T. Hu, Ralph Hull, H. G. Jacob, W. S. Jardetzky, R. V. Kadison, Shizuo Kakutani, Hyman Kamel, M. E. Kellar, D. E. Kibbey, J. F. Kiefer, H. S. Kieval, V. L. Klee, George Klein, Morris Kline, Daniel Kocan, E. R. Kolchin, B. O. Koopman, Alice Krikorian, J. C. Laffan, Serge Lang, C. E. Langenhop, J. A. Larrivee, Solomon Lefschetz, Marguerite Lehr, Benjamin Lepson, Howard Levi, M. A. Lipschutz, W. G. Lister, E. R. Lorch, Eugene Lukacs, Brockway McMillan, Wilhelm Magnus, E. W. Marchand, A. J. Maria, W. T. Martin, J. P. Mayberry, A. E. Meder, Jr., K. S. Miller, W. H. Mills, Don Mittleman, C. T. Molloy, Deane Montgomery, W. L. Murdock, C. A. Nelson, W. J. Nemerever, Morris Newman, J. A. Nohel, I. L. Novak, Adolf Nussbaum, A. F. O'Neill, H. A. Osborn, Robert Osserman, J. C. Oxtoby, L. E. Payne, H. O. Pollak, E. L. Post, G. N. Raney, H. E. Rauch, H. W. Raudenbush, Helene Reschovsky, Moses Richardson, J. E. Robinson, Robin Robinson, I. H. Rose, David Rosen, Maxwell Rosenlicht, J. E. Rosenthal, Walter Rudin, H. E. Salzer, Arthur Sard, Leo Sario, Samuel Schecter, Abraham Schwartz, I. E. Segal, H. N. Shapiro, Maurice Sion, O. K. Smith, P. A. Smith, H. H. Snyder, J. J. Sopka, I. A. Stegun, S. K. B. Stein, J. J. Stoker, J. T. Tate, R. L. Taylor, D. E. van Tijn, J. L. Tits, B. W. Volkmann, Alan Wayne, G. C. Webber, Alexander Weinstein, Louis Weisner, David Wellinger, A. B. Willcox, Otto Wolf, N. Z. Wolfsohn, Arthur Wouk, D. M. Young, L. A. Zadeh, J. A. Zilber, Leo Zippin.

Professor Alexander Weinstein of the University of Maryland delivered an invited address entitled *Generalized axially symmetric potential theory* at 2:00 P.M. Professor B. O. Koopman presided.

The morning session for contributed papers was presided over by Professor Ralph Hull. Professors Wilhelm Magnus and E. R. Lorch presided at the afternoon sessions.

Abstracts of the contributed papers are listed below, those with a "t" after their numbers having been read by title. Paper number 300 was presented by Dr. Young. Dr. Warga was introduced by Dr. Paul Brock.

ALGEBRA AND THEORY OF NUMBERS

268t. L. M. Blumenthal: *Two existence theorems for systems of linear inequalities.*

Let A denote an $m \times n$ matrix of rank r, A^T its transpose, and x a column matrix of n indeterminates. It is shown that the system of m linear inequalities $Ax \geqq 0$ (with at least one left-hand member positive) has a solution if and only if rows and corresponding columns of the matrix AA^T may be so shifted that (i) the upper left-hand principal minor M of order r is nonsingular and (ii) each rth order minor of AA^T formed from M by replacing its last row with that part of the jth row of AA^T contained in the first r columns ($j = r+1, r+2, \cdots, m$) is non-negative. It follows that the system of inequalities (in the wide sense)$Ax \geqq 0$ has a nontrivial solution if and only if whenever $r = n$, conditions (i), (ii) are satisfied. The theorems are consequences of a lemma in which the sides of the convexly metrized n-sphere determined by a great $(n-1)$-sphere are metrically characterized. (Received December 17, 1951.)

269. Volodymyr Bohun-Chudyniv: *Square matrices differing in some properties from Euler's squares.*

From the schemes solving Euler's problem proposed by: (1) L. Euler (Novi Comm. Acad. Petrop. vol. 15 (1770) p. 75; Comm. Arith. vol. 1, pp. 427–443); (2) C. Avery (Math. Misc., New York, 1839, p. 101); (3) G. K. Perkins (ibid., pp. 102–105); (4) V. A. Lebesque (Nouv. Ann. Math. vol. 15 (1856) pp. 403–407); (5) L. Bastien (Spinx-Oedipe vol. 7 (1912) p. 12); (6) N. Fuss (Mem. Acad. Sci., St. Petersburg vol. 4 (1813) pp. 240–247); just two types of square matrices are obtained. (1) Matrices which satisfy only 20 conditions of Euler, (2) those which satisfy all 22 conditions. The present paper gives other types of matrices which can be obtained from the author's scheme (1) solving Euler's problem (V. Bohun-Chudyniv, *Solution of Euler's problem*, Ukrainian Free Acad. of Sci., Regensburg, 1947), and not previously determined. These types are as follows: I. Matrices which satisfy besides Euler's 20 conditions also the six: (a) $A^2_{1,1}+A^2_{1,4}+A^2_{4,1}+A^2_{4,4}=A^2_{1,2}+A^2_{1,3}+A^2_{4,2}+A^2_{4,3}=A^2_{2,2}+A^2_{2,3}+A^2_{3,2}+A^2_{3,3}=A^2_{2,1}+A^2_{2,4}+A^2_{3,1}+A^2_{3,4}=S$. (b) $A_{1,1}A_{1,3}+A_{1,2}A_{1,4}+A_{4,1}A_{4,3}+A_{4,2}A_{4,4}=A_{2,1}A_{2,3}+A_{2,2}A_{2,4}+A_{3,1}A_{3,3}+A_{3,2}A_{3,4}=0$. Conditions (a) were noted also by Perkins (ref. 3 above). These matrices are obtained from the author's scheme (1) if its arguments satisfy the following relationships: $u_2=u_0t$, $u_3=u_0t$, $v_1=(2t/t^2-1)v_3$. Arbitrary values can be substituted for v_0, v_2, u_0, u_1, t. II. Matrices which satisfy 16, 14, 12, 8, 6, 4, and 2 conditions of Euler's problem and six conditions (a) and (b). These matrices are obtained from scheme (1) using relationships (2) on p. 4 of the author's paper mentioned above. (Received January 14, 1952.)

270t. Leonard Carlitz: *A divisibility property of the Bernoulli polynomials.*

Let u be integral (mod p) and put $\sigma_m(u) = p^{-r}(B_m(u)+1/p-1)$. It is proved that if $p \geqq 3$, $(p-1)p^r \mid m$, then $\sigma_m(n)$ is integral (mod p); indeed the residue (mod p) is specified. The corresponding result involving Bernoulli polynomials of higher order is also obtained. (Received December 24, 1951.)

271t. Leonard Carlitz: *Bernoulli numbers and polynomials of higher order.*

In the notation of Nörlund let $(x/(e^x-1)^2)^k e^{xu}=\sum_{m=0}^{\infty} B_m(u)x^m/m!$. Among the results of this paper the following may be mentioned. We assume $k<p-1$, $m\not\equiv 0, \cdots, k-1 \pmod{p-1}$, $m\geqq k\geqq 1$, and u integral $\pmod{p}$. 1. $B_m^{(k)}(u)$ is integral $\pmod{p}$. 2. If $p^r \mid (m)_k$, where $(m)_k=m(m-1)\cdots(m-k+1)$, then the numerator of $B_m^{(k)}$ is divisible by p^r. 3. If $(p-1)p^{e-1} \mid b, m\geqq rb+k$, then $\sum_{s=0}^{r}(-1)^{r-s}C_{r,s}T_{m+sb}^{(k)}\equiv 0 \pmod{p^{re}}$, where $T_m^{(k)}=B_m^{(k)}(u)/(m)_k$; also $\sum_{s=0}^{r}(-1)^{r-s}(C_{r,s}B_{m+sb}^{(k)}(u)\equiv 0 \pmod{p^{(r-k)e}}$. (Received December 24, 1951.)

272t. Leonard Carlitz: *Distribution of primitive roots in a finite field.*

Davenport (Quart. J. Math. Oxford Ser. vol. 8 (1937) pp. 308–312) proved that for large p one can always find a primitive root of the form $\theta+a$, where θ generates the field. He also showed that for given p, there exist fields such that no number $a\theta+b$ can be a primitive root. In the present note these results are extended in several directions. 1. Results are obtained concerning the number of numbers $\beta=\theta^r+a_1\theta^{r-1}+\cdots+a_r$ that belong to the exponent e, where $e \mid p^n-1$. 2. For fixed p and r, there exist fields such that no number β is a primitive root. (Received December 24, 1951.)

273t. Leonard Carlitz: *Primitive roots in a finite field.*

Ore (Trans. Amer. Math. Soc. vol. 36 (1934) pp. 243–274) has introduced the notion of a number of $GF(p^{nm})$ belonging to the exponent $a(x)$, where $a(x)$ *corresponds* to the ordinary polynomial $A(x)$, where $A(x) \mid x^m-1$. In particular β is a primitive root in this sense if it belongs to $x^{p^{nm}}-x$. It is natural to ask whether there exist numbers that are simultaneously primitive roots both in the ordinary and in Ore's sense. The principal result of this paper is the following. Let $e \mid p^{nm}-1$, $A(x) \mid x^m-1$. Then the number of numbers of $GF(p^{nm})$ belonging to e and $a(x)=\phi(e)\Phi(A)/p^{nm}+O(p^{nm(1/2+\epsilon)})$ $(p^{nm}\rightarrow\infty)$. (Received December 24, 1951.)

274t. Leonard Carlitz: *Sums of primitive roots in a finite field.*

Let $e_i \mid p^n-1$, $k_i \mid p^n-1$. Let β_i denote a number of $GF(p^n)$ that belongs to the exponent e_i, $i=1, \cdots, r$, and let ξ_i be an arbitrary number of $GF(p^n)$. We consider the following problems. 1. The number of solutions $\beta_1, \cdots, \beta_r$ of $\alpha=\alpha_1\beta_1+\cdots+\alpha_r\beta_r$, where $\alpha_i\neq 0$. 2. The number of solutions $\beta_1, \cdots, \beta_r, \zeta_1, \cdots, \zeta_s$ of $\alpha=\alpha_1\beta_1+\cdots+\alpha_r\beta_r+\delta_1\xi_1^{k_1}+\cdots+\delta_s\xi_s^{k_s}$. Asymptotic results are obtained for both problems. 3. The number of solutions $\beta, \xi_1, \cdots, \xi_{2s}$ of $\alpha=\beta_1+\delta_1\xi_1^2+\cdots+\delta_{2s}\xi_{2s}^2$. Here a simple exact formula is obtained. (Received January 7, 1952.)

275t. F. Marion Clarke: *Note on the effect of commutativity in a finite loop.*

It is noted that, as a direct consequence of the definition, loops of order less than or equal to 4 are commutative and associative; that a noncommutative loop of order 5 is not necessarily associative; but that commutativity implies associativity. It is shown by counter examples that in loops of higher order, prime or composite, commutativity alone does not produce associativity. If, however, every element of a commutative loop L_n of *any* order n has order n, and if L_n satisfies the weak associative law: $x\cdot[b\cdot(c\cdot b)]=[(x\cdot b)\cdot c]\cdot b$; then L_n satisfies the general associative law and L_n is a group. (Received January 10, 1952.)

276*t*. Anne C. Davis: *On order types whose squares are equal.*

The question of the existence of order types α and β such that $\alpha^2=\beta^2$ and $\alpha\neq\beta$ has been open. The answer turns out to be positive. Denumerable order types α and β satisfying the above conditions can be constructed as follows. Separate the set R of rationals into two disjoint sets S and T, both dense in R. Replace every element of S by an ordered set of type ω, leaving every element of T unchanged. Let α be the order type of the ordered set thus obtained and let $\beta=\alpha+\omega$. It is easily seen that $\alpha\neq\beta$, and, by an essential use of a theorem of Skolem (Skrifter utgit av Videnskapsselskapet i Kristiania, I. klasse 1920, no. 4, p. 32, Theorem 2) one can show that $\alpha^2=\beta^2=\alpha$. (Received February 4, 1952.)

277*t*. Chandler Davis: *Structure of certain Boolean algebras with operators.*

An S5 modalgebra is a Boolean algebra $\mathcal{B}$ with several S5 closure operators (McKinsey-Tarski) acting on it. Study of such systems is motivated by applications to logic, and because they provide generalizations of statements involving equivalence relations. (In case $\mathcal{B}$ is complete atomic and ϕ an equivalence relation among its points, assign to each $a\in\mathcal{B}$ the join a^x of all ϕ-equivalence classes including any point $\subseteq a$; this is an S5 closure operator.) Any projective algebra (Everett-Ulam) is evidently an S5 modalgebra under the operators $a\to a^x$ and $a\to a^y$ used in Chin and Tarski's axiom set (Bull. Amer. Math. Soc. Abstract 54-1-90); an S5 modalgebra can be obtained similarly from any projective algebra *relativized* to an arbitrary fixed element. Theorem: Every S5 modalgebra with two operators is obtained in this way. Generalization to k operators is immediate. This theorem may be thought of as asserting "co-ordinatizability" of these systems. (Received January 3, 1952.)

278. Beno Eckmann: *Cohomology of groups and transfer.*

For an arbitrary group A and an A-module J (an Abelian group, written additively, with A as group of left operators) let $H^p(A, J)$, $p=1, 2, \cdots$, be the (equivariant) cohomology groups of A with coefficient group J. For any subgroup B of A there is a coefficient group J' such that $H^p(B, J)$ is naturally isomorphic to $H^p(A, J')$; namely, $J'=A$-module of all functions ψ from A to J such that $\psi(ba)=b\cdot\psi(a)$ for all $b\in B$, $a\in A$. An A-homomorphism f of J' into J induces therefore homomorphisms S_f^p of $H^p(B, J)$ into $H^p(A, J)$, for all p.—Special case: If the index n of B in A is finite, let $a_1, \cdots, a_n$ be any representatives of the left cosets and define f by $f(\psi)=\sum a_i^{-1}\psi(a_i)$. The induced homomorphisms $S_f^p=T^p$ of the cohomology groups generalize the "transfer of A into B": If A operates trivially in J, the homomorphism T^1 of $H^1(B, J)$ $=\text{Hom}\,(B, J)$ into $\text{Hom}\,(A, J)$ is dual to the transfer. Properties analogous to those of the transfer hold for all the T^p. E.g., let I^p denote the homomorphism of $H^p(A, J)$ into $H^p(B, J)$ induced by the injection of B into A; then $T^p(I^ph)=nh$ for all $h\in H^p(A, J)$. (Received January 24, 1952.)

279. Trevor Evans: *Embeddability and the word problem.*

A method exists for solving the word problem in an equationally defined class of abstract algebras if, and only if, a method exists for deciding whether an incomplete algebra can be embedded. This includes the previous result of the author (*The word problem for abstract algebras*, J. London Math. Soc. vol. 26 (1951)) that the word problem can be solved for a class of algebras having the property that any incomplete algebra can be embedded. Using the methods of that paper it can be assumed that, in solving the word problem, the two words to be compared are generators and that the generators and relations form an incomplete algebra. If the two generators are not equivalent, by equating some of the generators, an embeddable incomplete algebra can be found containing the two generators as distinct elements. If the two generators are equivalent no such incomplete algebra exists. On the other hand, if the word problem can be solved, the embeddability problem can also be solved since an incomplete algebra can be embedded only if its elements are not equivalent in the algebra it freely generates. (Received January 8, 1952.)

280*t*. W. H. Gottschalk: *The extremum law.*

The extremum law is defined to be the class E of all statements S such that S is equivalent to the axiom of choice and such that the conclusion of S asserts the existence of an extremal (maximal or minimal) element. A member of E is called a form of the extremum law. Various forms of the extremum law are known under the names of Zorn's lemma, Hausdorff maximality principle, etc. The purpose of this note is to point out a particularly simple form of the extremum law, namely: If R is a binary relation, then there exists a maximal set A such that $A \times A \subset R$. (Received December 26, 1951.)

281*t*. I. N. Herstein: *A generalization of a theorem of Jacobson*. II.

In his paper *A generalization of a theorem of Jacobson* (Amer. J. Math. vol. 73 (1951) pp. 756–762) the author proved the following theorem: Let R be a ring with center Z such that $x^n - x \in Z$ for all $x \in R$, n a fixed integer larger than 1; then R is commutative. In this paper this result is extended to the case where $n = n(x)$ is no longer fixed but an integer depending on x, where, however, we assume that $n(x) < N$ for all $x \in R$. The methods of the previous paper, together with several applications of the "Dirichlet box principle" yield the result. (Received January 11, 1952.)

282*t*. I. N. Herstein: *Finite multiplicative subgroups in division rings.*

This paper is concerned with extending the result that any finite multiplicative subgroup of the multiplicative group of a commutative field is cyclic. The following results are obtained: (1) Let K be a division ring of characteristic $p \neq 0$. Then any finite multiplicative subgroup in K is cyclic. (2) Let K be any division ring. Then any multiplicative, odd-ordered subgroup in K is metacyclic. (3) Let Q be the quaternions over the reals. Then multiplicative subgroups of odd order in K must be cyclic. It is conjectured that any multiplicative subgroup of odd order in a division ring must be cyclic. (Received January 11, 1952.)

283. Ralph Hull: *Cyclotomy and cyclic fields.*

For $e = 3, 4, 5, 6, 8, 12$, all cyclic fields of degree e are determined by exhibiting the equations satisfied by the respective periods of roots of unity which generate the fields. The coefficients of the equations are expressed in terms of the conductor c of the field and its factors, and representations of c or one of its factors by certain quadratic forms. For example, when $e=8$, the quadratic forms are x^2+4y^2 and a^2+2b^2. (Received January 11, 1952.)

284. Serge Lang: *On quasi algebraic closure.*

A field F is defined to be C_i if every form with coefficients in F in n variables, of degree d, and $n>d^i$, has a nontrivial zero in F. For $i=1$ this is the quasi algebraic closure of Artin. We prove that a function field in s variables over a C_i constant field is C_{i+s}, and also some conjectures of Artin concerning C_1 and C_2 fields. In particular, the power series field over a finite field is C_2 and the following fields are C_1: A field F complete under a discrete valuation with algebraically closed residue class field, and the maximal unramified extension of a field with a discrete valuation and perfect residue class field. In fact any subfield F_0 of F which is dense and algebraically closed in F is C_1. These results can be applied to local class field theory to prove Chevalley's theorem that over a p-adic field a 2-cocycle of period r is split by every field of degree r. (Received January 10, 1952.)

285t. D. J. Lewis: *Norm groups of certain simple algebras.*

The principal result obtained is that the reduced norm group of a central simple algebra of order nine over an algebraic number field is the multiplicative group of that field. The proof utilizes the fact that a central simple algebra of order nine is either a total matrix algebra or a division algebra. In either case the reduced norm function is a homogeneous polynomial of degree three in nine indeterminates. Use is then made of the author's result on cubic homogeneous polynomials over p-adic fields and the Maass, Hasse, Schilling norm theorem. A somewhat similar result is obtained for central simple algebras of order four. (Received January 10, 1952.)

286. Maxwell Rosenlicht: *Differentials of second kind for algebraic function fields of one variable.*

Let K be a field of algebraic functions of one variable over the constant field k, which, for simplicity, we assume algebraically closed. The differential ω of K is said to be of the second kind if for each place P of K/k there exists a function $f_P \in K$ such that $\omega - df_P$ is finite at P. The differentials of second kind form a vector space over k that contains the space of exact differentials. The quotient space is known to have dimension $2g$, where g is the genus of K, in case the characteristic of K is zero; if the characteristic of K is $p\neq 0$, the dimension is g. If a differential ω can be approximated arbitrarily closely at a certain fixed place P by exact differentials, then, in characteristic zero, $\mathrm{res}_P\, \omega=0$; if the characteristic is $p\neq 0$, ω is exact. (Received January 10, 1952.)

287t. A. R. Schweitzer: *An identity in Grassmann's extensive algebra.*

Let $E_1, E_2, \cdots, E_n$ $(n=2, 3, \cdots)$ be mutually perpendicular unit vectors in Grassmann n-space. If $X_n=\sum_{i=1}^n x_iE_i$ and $Q(X_nE_i)=\sum_{j=1}^n x_jQ(E_jE_i)$ $(i=1, 2, \cdots, n)$ where $Q(X_nE_i)$ and $Q(E_jE_i)$ denote operators on E_i which transform E_i into X_n and E_j respectively, then the following expression is an identity provided $Q(E_iE_i)=1$ and $Q(E_iE_j)=-Q(E_jE_i)$ $(i\neq j)$: $\sum_{i=1}^n x_i^2=\sum_{i=1}^n x_iQ(X_nE_i)$. This identity permits the ready calculation of $Q(Y_nX_n)$ where the latter symbol is an operator on X_n which transforms X_n into the vector $Y_n=\sum_{i=1}^n y_iE_i$ as is easily verified by multiplying the above identity on both sides by $Q(Y_nX_n)$ and noting that

$Q(Y_nX_n)\times Q(X_nE_i)=Q(Y_nE_i)$. Reference is made to the author's article in Math. Ann. vol. 69 (1910). (Received January 9, 1952.)

288. Bodo Volkmann: *On classes of sets of integers.*

For any infinite set $\mathfrak{A}=\{a_1, a_2, \cdots\}$ of positive integers the dyadic value ("Dualwert") is defined to be $\Gamma(\mathfrak{A})=\sum_{a\in\mathfrak{A}}2^{-a}$. The author (in a paper to be published in the J. Reine Angew. Math.) has investigated classes $N=\{\mathfrak{A}, \mathfrak{B}, \cdots\}$ of such sets and applied Hausdorff's measure theory to the corresponding point sets $\Gamma N=\{\Gamma(\mathfrak{A}), \Gamma(\mathfrak{B}), \cdots\}$. Based on Knichal's and Besicovitch's methods, some general theorems on Hausdorff dimensions are obtained by means of which dim ΓN is determined in the cases when N is (a) the class of infinite subsets of a fixed set $\mathfrak{A}$; (b) the class of all sets $\mathfrak{A}$ for which a given number $\zeta(0\leqq\zeta\leqq I)$ lies between their lower and upper asymptotic densities. The corresponding results are: (a) dim ΓN = lower asymptotic density of $\mathfrak{A}$ (containing the proof of a conjecture due to E. F. and R. C. Buck, Amer. J. Math. vol. 69, pp. 413–420); (b) dim $\Gamma N=\log\gamma_i/\log 2$, γ_i being the positive root of the equation $\xi^i-\xi^{i-1}-\cdots-\xi^0=0$; (c) dim $\Gamma N(\zeta\log\zeta+(1-\zeta)\log(1-\zeta))/\log(1/2)$. (Received January 9, 1952.)

Analysis

289. Joseph Andrushkiw: *On the zeros of integral functions.*

Let the zeros of the polynomial $f(z)$ be pure imaginary. Applying a theorem of Hurwitz (Math. Ann. vol. 46, *Ueber die Bedingungen, unter welchen eine Gleichung nur Wurzeln mit negativen reellen Teilen besitzt*) to the polynomial $F(z)=f(z+x)$, $x\neq 0$, one obtains sufficient condition that a polynomial with real coefficients have all its zeros pure imaginary: All principal minors of even order of a determinant of $2n$th order, whose elements are the coefficients of the polynomial, are positive. Similar sufficient conditions can be deduced for a polynomial with all positive or all negative zeros. By proper transformation of the above determinants A_j they are shown to be symmetric functions of the zeros and the sufficient conditions appear to be also necessary. Let n_1 and n_2 be the numbers of sign variations in the sequences 1, A_k/A_{k+1} and $\overline{A}_k/\overline{A}_{k+1}$, $1\leqq k\leqq 2n-1$, where $\overline{A}_j=(-1)^{C_{j+1,3}}A_j\neq 0$, respectively. Then the number of negative zeros of a polynomial is $n-n_1$, and the number of positive zeros $n-n_2$; $n_1+n_2=n$ is the necessary and sufficient condition that all zeros be real and distinct. The same methods applied to the results of Grommer (J. Reine Angew. Math. vol. 144, *Ganze transzendente Funktionen mit lauter reellen Wurzeln*) extend the criteria to some types of integral transcendental functions. (Received December 31, 1951.)

290*t*. S. D. Bernardi: *A critical evaluation of the classical and variational methods in the theory of schlicht functions.*

The paper consists primarily of a critical review of the classical results in the theory of functions $f(z)=z+a_2z^2+a_3z^3+\cdots$ regular and schlicht in the unit circle $|z|<1$. It is suggested that the "area-principle" of Gronwall be generalized by the introduction of proper weight functions. It is also suggested that an improvement of the inequality $\lim_{n\to\infty}|a_n|/n\leqq e/2$ might possibly result from the generalized form of the area-principle due to Prawitz by the application of an induction process. An example is given showing the independence of the inequality of Prawitz from the parametric representations of the coefficients a_2 and a_3 given by A. C. Schaeffer and D. C. Spencer. It is also shown that if a_n are all real and $f(z)$ is convex in the direction of the imaginary axis, then $|a_n| \leqq 1$. The proof of this last result follows readily since, under the above conditions, $t(z) = zf'(z)$ is typically-real for $|z| < 1$ and hence $p(z) = (1-z^2)t(z)/z = 1 + \sum_1^\infty b_n z^n$, $b_n = (n+1)a_{n+1} - (n-1)a_{n-1}$, is a function of positive real part in $|z| < 1$ so that $|b_n| \leqq 2$, from which the result follows. (Received January 9, 1952.)

291*t*. R. P. Boas: *Integrability along a line for a class of entire functions.*

Duffin and Schaeffer have shown in a paper presented to the International Congress of Mathematicians in 1950 that if $f(z)$ is an entire function of exponential type $c < \pi$, then $\int_{-\infty}^{\infty} |f(x)|^2 dx \leqq K \sum_{-\infty}^{\infty} |f(\lambda_n)|^2$ provided $|\lambda_n - n| \leqq L$ and $|\lambda_{n+m} - \lambda_n| \geqq \delta > 0$, $m \neq 0$. This is generalized to $\int_{-\infty}^{\infty} \phi(H|f(x)|)dx \leqq K \sum_{-\infty}^{\infty} \phi(|f(\lambda_n)|)$ for a class of functions which includes $\phi(t) = t^p$, $p > 0$, and all increasing positive convex functions. The proof depends on an interpolation series. (Received December 14, 1951.)

292. F. E. Browder: *The Dirichlet problem for linear elliptic differential equations of even order with variable coefficients. The Fredholm alternative.*

The Dirichlet problem for a linear elliptic differential operator K of order $2m$ on a bounded domain D of E^n is the following: Given $h \in C^1(D) \cap L^2(D)$, $g \in C^m(D)$ such that the mth derivatives of g are square-summable, to find $u \in C^{2m}(D)$ for which $Ku = h$ and such that $u - g$ and its derivatives of order not greater than $m-1$ vanish on the boundary of D in a suitable sense. Suppose that the coefficients of the jth derivatives in K belong to $C^{2m+j}(D)$ and the mth derivatives of these coefficients are continuous on the closure of D. It is shown that under these conditions if the solution of the Dirichlet problem is unique, then it always exists and conversely. In the general case $Ku = h$ has a solution for zero boundary values for all h such that $\int_D hw = 0$ for all w in the finite-dimensional set of null solutions of the adjoint equation $\overline{K}w = 0$. The dimension of the set of null solutions for K is equal to that for $\overline{K}$. (Received January 9, 1952.)

293*t*. F. E. Browder: *The eigenvalues and eigenfunctions of the Dirichlet problem for self-adjoint linear elliptic equations.*

If L is a linear elliptic differential operator on a bounded domain D in E^n, A a linear differential operator of order less than $2m$, the order of L, then $w \in C^{2m}(D)$ is an eigenfunction of L with respect to A with eigenvalue k if $Lw = kw$ and w together with all its derivatives of order less than m vanishes on the boundary of D in a suitable sense. Under differentiability conditions on the coefficients of L and A, a complete theory of such eigenfunctions and eigenvalues is established. In particular it is shown that if L is positive self-adjoint and A is self-adjoint, then the eigenfunctions of L with respect to A are complete in the space of functions vanishing on neighborhoods of the boundary of D. (Received January 9, 1952.)

294*t*. R. M. Conkling and D. O. Ellis: *On metric groupoids.*

A single operation algebra (groupoid) which forms a metric space is called a *metric groupoid* if the operation is simultaneously continuous in the metric topology. It is called a *uniformly metric groupoid* (u.m.g.) if this simultaneous continuity is simultaneously uniform. Certain algebraic properties of subsets of metric groupoids are shown to extend to their topological closures. Among these are commutativity, associativity, identity element, ideal property, and subgroupoid property. It is shown that the Cantor-Hausdorff completion of a u.m.g. may be made into a u.m.g. in which the original u.m.g. is isometrically isomorphically imbedded. In particular, it is shown that a uniformly metric group is a topological group and that any uniformly metric (Abelian) group may be isometrically isomorphically imbedded in a metrically complete uniformly metric (Abelian) group. (Received December 17, 1951.)

295. R. B. Davis: *A boundary value problem for third-order linear partial differential equations of composite type.*

It is proved that, by suitable changes of variables, all third-order partial differential equations of composite type (i.e., having exactly one real characteristic), in which the third-order derivatives occur linearly and with continuously differentiable coefficients, can be reduced to a stated canonical form. Using this canonical form, the following boundary value problem for a certain linear composite equation $L(u)=f$ with coefficients belonging to C^1 is solved in the sense described: for a region for which the classical Green's function exists and satisfies certain inequalities, boundary values belonging to C^3 are prescribed on the entire boundary for the derivative of u along the characteristics, and are prescribed along a suitable curve for u itself. It is then proved that either the resulting boundary value problem has a unique solution, or else the homogeneous problem obtained by prescribing zero boundary values for $L(u)=0$ has a nontrivial solution. Two cases are established in which the second alternative is impossible. The methods employed are strongly analogous to those used in the study of second-order elliptic equations. (Received January 8, 1952.)

296. J. W. Green: *Approximately convex and approximately subharmonic functions.*

Hyers and Ulam have defined (Bull. Amer. Math. Soc. Abstract 57-4-380) an ϵ-convex function to be a function f in a convex domain of Euclidean n-space satisfying the inequality $f(hx+(1-h)y)\leqq\epsilon+hf(x)+(1-h)f(y)$ for $0\leqq h\leqq 1$. They proved the theorem: If f is continuous and ϵ-convex, there exists a convex function differing from f by not more than $k_n\epsilon$, where $2k_n=1+(n-1)(n+2)/2(n+1)$. In this paper, this theorem is proved in a different manner with an improved value of $2k_n\sim\log_2(n+1)$, and only slightly poorer results obtained for discontinuous f. Sharpness of the inequality for small n is shown, and some properties of the extreme functions discussed. An ϵ-subharmonic function is defined to be an upper semicontinuous function f such that if h is a harmonic function dominating f on the boundary of a domain, then $h+\epsilon$ dominates f inside the domain. It is proved that if f is ϵ-subharmonic, there exists a subharmonic function u, namely the greatest subharmonic nunorant of f, such that $u\leqq f\leqq u+\epsilon$. (Received December 14, 1951.)

297. Henry Helson: *Isomorphisms of abelian group algebras.*

Let G and H be locally compact abelian groups with group algebras $L(G)$ and $L(H)$ respectively. Let T be a linear operator mapping $L(G)$ onto $L(H)$ isomorphically. Then G and H are isomorphic groups, under the following hypotheses: the dual group of G (or of H) is connected, and T has bound less than two. The proof depends on a method found by A. Beurling and applied by him to a related problem on the real line. The conclusion of the theorem holds without restriction on the groups if the bound of T is one. (This result has been extended by J. G. Wendel to nonabelian group algebras.) (Received January 7, 1952.)

298*t*. M. L. Stein: *Sufficient conditions for the convergence of Newton's method in complex Banach spaces.*

Let $T(y)$ be an operator defined on one Banach space into another and let $\delta T(y; h)$ be its first variation. The Banach space analogue of Newton's method is given by the iteration formula $(*)$ $y_{i+1} = y_i - \delta T^{-1}(y_i; T(y_i))$, $i = 0, 1, 2, \cdots$, where δT^{-1} denotes the inverse with respect to h of δT. Kantorovič [Uspehi Matematičeskih Nauk N.S. vol. 3 (1948) pp. 1237–1240] has established conditions under which a sequence $\{y_i\}$ determined by $(*)$ will converge to a solution of $T(y) = 0$ in a real Banach space. The present paper establishes the following convergence theorem for complex Banach spaces: Let Y and Z be complex Banach spaces and let $T(y)$ be defined on $S: \|y\| < \rho$, $\rho > 0$, into Z. Let the following conditions hold: (i) $T(y)$ is G-differentiable, (ii) there exists y_0 in S, a positive number a, and a sufficiently small positive constant M such that $\|T(y)\| \leqq M$ if $\|y - y_0\| \leqq a$, (iii) $\delta T(y_0; h)$ is a 1–1 mapping of a subspace of Y onto a subspace of Z. Then the sequence $\{y_i\}$ given by $(*)$ is well defined and there exists $d > 0$ such that $\{y_i\}$ converges quadratically to a unique element $\bar{y}$ of the d-neighborhood of y_0. Furthermore $\bar{y}$ is the unique solution of $T(y) = 0$ in this neighborhood. (Received December 17, 1951.)

299*t*. J. L. Walsh: *Degree of approximation to functions on a Jordan curve.*

Let C be an analytic Jordan curve contained in an annular region D. Let the sequence of functions $F_n(z)$ analytic in D satisfy the inequality $|F_n(z)| \leqq AR^n$ in D and for some $f(z)$ the inequality $|f(z) - F_n(z)| \leqq A_0/n^{p+\alpha}$ on C. Then on C the function $f(z)$ is of class $L(p, \alpha)$ if $0 < \alpha < 1$ and of class Z_p if $\alpha = 1$. For the respective components $F_{1n}(z)$ and $F_{2n}(z)$ of $F_n(z)$ and $f_1(z)$ and $f_2(z)$ of $f(z)$ one has $|f_j(z) - F_{jn}(z)| \leqq A_1 \log n/n^{p+\alpha}$ on C, $j = 1, 2$. In particular this conclusion holds if $z = 0$ lies interior to C and if $F_n(z) = F_{1n}(z) + F_{2n}(z)$, $F_{1n}(z) = \sum_0^n a_{nk}z^k$, $F_{2n}(z) = \sum_{-n}^{-1} a_{nk}z^k$. (Received January 29, 1952.)

300. J. L. Walsh and D. M. Young: *A new bound for the moduli of continuity of harmonic functions.*

The use of the analogue for harmonic functions of Milloux's theorem for analytic functions shows that if R is an arbitrary bounded simply connected region, if the modulus of continuity of $u(x, y)$ on the boundary of R is $\omega(\delta)$, if M is the oscillation of $u(x, y)$ on the boundary, and if $u(x, y)$ is harmonic in R and continuous in the closure $\bar{R}$ of R, then the modulus of continuity $\omega^*(\delta)$ of $u(x, y)$ in $\bar{R}$ satisfies the inequality $\omega^*(\delta) \leqq (D(\delta/D)^{1/3}) + (8M/\pi)(\delta/D)^{1/3}$, where D is any positive constant. A sharper inequality which is valid under stronger assumptions on R has been presented by the authors (Bull. Amer. Math. Soc. Abstract 58-2-206). (Received January 11, 1952.)

301. Jack Warga: *On a class of iterative procedures for solving normal systems of differential equations.* Preliminary report.

Given a system of differential equations $z' = f(z, t)$, $z(t_0) = z_0$, where z and $f(z, t)$

are n-dimensional vectors, a sequence of vector functions $y_j(t)$ is defined by $y_j' = G_j(y_j, t)$, $y_j(t_0) = z_0$. The vector functions $G_j(u, t)$ are bounded and continuous in some domain D of the u, t space, satisfying a Lipschitz condition in u and $f(y_j(t), t) - G_{j+1}(y_j(t), t) \to 0$ uniformly in t. A proof is given of the convergence of the sequence $y_j(t)$ to the solution of the given system and a study is made of convergence properties in particular cases. Various iterative procedures described in the literature are shown to be of the type under discussion. New procedures are suggested which are applicable to numerical work and practical calculations. (Received January 8, 1952.)

302*t*. Bertram Yood: *Difference algebras of linear transformations on a Banach space.*

Let $E(X)$ be the algebra of all bounded linear transformations on an infinite-dimensional Banach space X into X. Let $K(X)$ be the subset of all completely continuous transformations in $E(X)$. Let π be the natural homomorphism of $E(X)$ onto the difference algebra $E(X) - K(X)$. Let G be the set of regular elements of $E(X) - K(X)$ and let R be its radical. It is shown that if $T \in \pi^{-1}(G)$, $U \in \pi^{-1}(R)$, then T, $T+U$, T^*, and T^*+U^* have finite nullity and nul $(T^*) -$ nul $(T) =$ nul $(T^*+U^*) -$ nul $(T+U)$. For $T = I$, $U \in K(X)$ this reduces to a well known result of Schauder, Studia Math. vol. 2 (1930). The relations between the sets of (left, right) regular elements in $E(X)$ and in $E(X) - K(X)$ are investigated in some detail. (Received January 9, 1952.)

Applied Mathematics

303. Abolghassem Ghaffari: *The behavior of hodograph functions for slow motion.*

In a recent paper (Bull. Amer. Math. Soc. Abstract 58-2-112) the author showed that the elementary solutions of the hodograph equations depend on the functions (1) $\psi_m(\zeta) = \zeta^{m/2} F(a_m, b_m, c_m; \zeta)$ for $m > 0$, $\psi_{-n}(\zeta) = [\zeta^{-n/2} - K\zeta^{n/2}\zeta_s^{-n}][1 + O(\zeta)]$ for $m = -n < 0$, and $\phi_m(\zeta) = P/m\partial\psi_m/\partial\zeta$ for all values of m, where $\zeta = w^2/w_m^2$ (w being the compressible fluid speed with w_m its maximum, ζ_s being the value of ζ at the sonic speed, K being independent of ζ, F being the hypergeometric function formed with a_m, b_m, and c_m, and P being $2\zeta(1-\zeta)^{-\beta}$ where $\beta = (\gamma - 1)^{-1}$ with γ the adiabatic index). The functions (1) which shall be referred to as *hodograph functions* can be written in the form (2) $\psi_m(\zeta) = \zeta^{m/2} + O(\zeta^{m/2+1})$, $\phi_m(\zeta) = \zeta^{m/2} + O(\zeta^{m/2+1})$, for all values of m. Since ϕ_m and ψ_m behave like $\zeta^{m/2}$ for small ζ, as shown in (2), it follows that when $m > 0$, the hodograph functions are both positive monotone increasing functions of ζ and their first derivatives are positive near $\zeta = 0$. Hence they are all positive throughout the subsonic range. It can be shown that when $m < 0$, ψ_m is positive, but this is not true of its derivative or of ϕ_m. (Received January 7, 1952.)

304*t*. R. L. Sternberg: *A criterion for stability of nonlinear pulsed systems.*

Consider a pulsed system for which the output sequence $\{y_n\}$ is related to the input sequence $\{x_n\}$ by a relation of the form $y_n \equiv \sum_{k=1}^{\infty} w_k(x_{n-k})$ ($n = 0, \pm 1, \pm 2, \cdots$) where the $w_k(x)$ are arbitrary real single finite-valued functions of a real variable x. Call the system stable if $\{x_n\}$ bounded implies that $\{y_n\}$ is defined and bounded. Then by generalizing a method given by W. Hurewicz the following theorem is obtained. If the system is stable, then for each m one has ($*$) $\left| \sum_{k=1}^{\infty} w_k[m \operatorname{sgn} w_k(m)] \right| \neq \infty$, i.e., for each m the series in ($*$) converges or oscillates; moreover, in case for each $a>0$ there exists an m_a such that on $[-a, a]$ one has $0\leqq w_k(x)\leqq w_k(m_a)$ $(k=1, 2, \cdots)$, then the system is stable if and only if for each m the series in $(*)$ converges. This work was supported in part by a contract with the Rome Air Development Center of the U. S. A. F. (Received January 11, 1952.)

Geometry

305t. Edward Kasner and Don Mittleman: *Strongly orthogonal algebraic curves.*

If p is a point of intersection of two algebraic curves $C_1:\phi=0$ and $C_2:\psi=0$ at which all of the branches of C_1 are orthogonal to all the branches of C_2, then C_1 and C_2 are called totally orthogonal at p. Two algebraic curves which are totally orthogonal at each finite point of intersection and for which, at each point of intersection which lies on the infinite line, the order of $\omega\equiv\phi_x\psi_x+\phi_y\psi_y$ is at least r_i+s_i-1 are said to be strongly orthogonal. The fundamental theorem of M. Nöther is shown to be applicable and ω is a linear combination of ϕ and ψ. Necessary and sufficient conditions that two algebraic curves be strongly orthogonal are given (a) when one of the curves is a straight line, the other being arbitrary, and (b) when each of the two curves is a nondegenerate conic. (Received January 8, 1952.)

306. V. L. Klee: *The critical set of a convex body.*

For a convex body C in E^n, $x\in\text{Int } C$, and $y\in FC$, let $\rho(y, x)=yx/yp$, where $[y, p]$ is the chord containing $[y, x]$. Let $\rho(x)=\sup_{y\in FC}\rho(y, x)$, $S(x)=\{y\mid y\in FC$ and $\rho(y, x)=\rho(x)\}$, $r=\inf_{x\in\text{Int } C}\rho(x)$, and $C^*=\{x\mid \rho(x)=r\}$. C^* and r are the critical set and critical ratio investigated in E^2 by B. H. Neumann [J. London Math. Soc. vol. 14 (1939) pp. 262–272] and in E^n by Hammer and Sobczyk [Bull. Amer. Math. Soc. Abstract 57-2-112]. This paper provides a more detailed discussion in E^n. Typical results (with $C\subset E^n$, $n\geqq 2$) are (1) dim $C^*\leqq n-2$; (2) for each $x\in C^*$, $S(x)$ contains at least three points; (3) $r(1-r)^{-1}+\dim C^*\leqq n$, and if greater than $n-1$, then for no $x\in C^*$ is $S(x)$ contained in an open half-space parallel to C^* but not containing C^*. Critical sets of certain special bodies are investigated in detail. One of the principal tools is an extension of Helly's theorem on the intersection of convex sets. (Received January 16, 1952.)

307t. T. K. Pan: *Pseudo normal curvature and related curves.*

Let v be the vector of a vector field in a surface of ordinary space. The derived vector of v along a curve of the surface can be decomposed into a component tangential and a component normal to the surface. The former component called the angular spread has been studied by W. C. Graustein and R. M. Peters. It is the latter component, which is evidently a generalization of the normal curvature vector, with which the paper is concerned. Pseudo normal curvature, pseudo asymptotic line, pseudo line of curvature, indicatrix and curve of a vector field in a surface are defined, new characteristics of principal curvature and line of curvature are derived, and most of the theorems connected with normal curvatures, such as Meusnier's theorem, Euler's theorem, etc., are generalized first for a surface in an ordinary space and then extended to V_n in V_{n+1} and to V_n in V_m. (Received January 8, 1952.)

308. Helene Reschovsky: *Sets associated with convex bodies.*

Let $D(k, n)$, $1/2\leqq k<1$, $n=0, 1, 2, \cdots$, be set of points within a plane convex body such that there exist exactly n chords of C which are divided by these points in the ratio k, the ratio of the larger segment to the whole chord. B. H. Neumann (J. London Math. Soc. vol. 14) has proved that there exists in the interior of a plane convex body a unique point such that the maximum ratio in which a point divides all chords passing through it assumes a minimum value r^*, $1/2 \leqq r^* \leqq 2/3$. In this note it is proved that (1) $D(k, 0)$ is two-dimensional when $k > r^*$ and empty when $k \leqq r^*$, (2) $D(k, n)$, $n \neq 0$, is two-dimensional for an even number n and one-dimensional for an odd number n, when $k > 1/2$, (3) $D(1/2, n)$, $n \neq 0$, is two-dimensional for an odd number n and one-dimensional for an even n. (Received January 4, 1952.)

309*t*. A. R. Schweitzer: *On the derivation of the regressive product in Grassmann's geometrical calculus.* III.

Values are derived for the regressive products $P_3(\alpha_1\alpha_2\alpha_3\alpha_4) = \alpha_1\alpha_2 \cdot \alpha_1\alpha_3\alpha_4$ and $P_3'(\alpha_1\alpha_2\alpha_3\alpha_4) = \alpha_1\alpha_3\alpha_4 \cdot \alpha_1\alpha_2$ (Grassmann, *Gesammelte Werke*, vol. 1, part 1, pp. 231, 244). From $P_1(\alpha_1\alpha_2\alpha_3\alpha_4) = c_1 q \cdot \alpha_1\alpha_2$ and $P_2(\alpha_1\alpha_2\alpha_3\alpha_4) = c_2 q^2 \cdot \alpha_1$ since the product $P_2(\alpha_1\alpha_2\alpha_3\alpha_4)$ is associative, one obtains $c_2 q^2 \cdot \alpha_1 = c_1 q \cdot \alpha_1\alpha_2 \cdot \alpha_1\alpha_3\alpha_4$ and (by a change of notation) $c_2 q^2 \cdot \alpha_1 = \alpha_1\alpha_3\alpha_4 \cdot c_1 q \cdot \alpha_1\alpha_2$. It is assumed that the latter statements are respectively equivalent to $c_2 q \cdot \alpha_1 = c_1\alpha_1\alpha_2 \cdot \alpha_1\alpha_3\alpha_4$ and $c_2 q \cdot \alpha_1 = \alpha_1\alpha_3\alpha_4 \cdot c_1 \cdot \alpha_1\alpha_2$ and that $c_1 = c_2 = 1$. Definitions: $[\alpha_1\alpha_2\alpha_3\alpha_4] \cdot \alpha_1 = \alpha_1\alpha_2 \cdot \alpha_1\alpha_3\alpha_4$ means: There exist $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$ such that $q \cdot \alpha_1 = \alpha_1\alpha_2 \cdot \alpha_1\alpha_3\alpha_4$ and analogously for $[\alpha_1\alpha_2\alpha_3\alpha_4] \cdot \alpha_1 = \alpha_1\alpha_3\alpha_4 \cdot \alpha_1\alpha_2$. Therefore $\alpha_1\alpha_2 \cdot \alpha_1\alpha_3\alpha_4 = \alpha_1\alpha_3\alpha_4 \cdot \alpha_1\alpha_2 = [\alpha_1\alpha_2\alpha_3\alpha_4] \cdot \alpha_1$. (Received January 9, 1952.)

310*t*. A. R. Schweitzer: On the derivation of the regressive product in Grassmann's geometrical calculus. IV.

The complete statements governing the regressive products $P_1(\alpha_1\alpha_2\alpha_3\alpha_4) = \alpha_1\alpha_2\alpha_3 \cdot \alpha_1\alpha_2\alpha_4$, $P_3(\alpha_1\alpha_2\alpha_3\alpha_4) = \alpha_1\alpha_2 \cdot \alpha_1\alpha_3\alpha_4$, $P_3'(\alpha_1\alpha_2\alpha_3\alpha_4) = \alpha_1\alpha_3\alpha_4 \cdot \alpha_1\alpha_2$ are represented by the abstract statement $A \cdot CD = AD \cdot C$ where for P_1, $A = \alpha_1\alpha_2\alpha_3$, $C = \alpha_1\alpha_2$, $D = \alpha_4$; for P_3, $A = \alpha_1\alpha_2$, $C = \alpha_1$, $D = \alpha_3\alpha_4$; and for P_3', $A = \alpha_1\alpha_3\alpha_4$, $C = \alpha_1$, $D = \alpha_2$. Reference is made to Grassmann, *Gesammelte Werke*, vol. 1, part 1, pp. 213–219, 295–296. (Received January 9, 1952.)

Logic and Foundations

311*t*. Hyman Kamel: *Relational algebra.*

A new system of axioms is given for a relational algebra $\mathfrak{A} = (\mathfrak{C}, ', \cap, \cup, ^{\cup}, \cdot, \Lambda, V, I)$, namely: (1) $(\mathfrak{C}, ', \cap, \cup, \Lambda, V)$ is a boolean algebra, (2) $(RS)T = R(ST)$, (3) $RI = R$, (4) $R^{\cup\cup} = R$, (5) $(RS)^{\cup} = S^{\cup}R^{\cup}$, (6) $R\Lambda = \Lambda = \Lambda R$, (7) $RS \cap T \subset (R \cap TS^{\cup})(S \cap R^{\cup}T)$. The axiom system is proved equivalent to Tarski's (Journal of Symbolic Logic (1941) pp. 76–77). A proof is given that $\mathfrak{A}$ is simple ($RV = V$ or $VR' = R$) if and only if $R \neq \Lambda$ implies $VRV = V$. The bulk of the identities in Schroder's book are deduced. The notion of a uniform relational algebra (u.r.a.) is introduced by postulating a subclass $\mathfrak{U} = [\alpha, \beta, \gamma, \cdots]$ of $\mathfrak{A}$ such that: (1) $\mathfrak{U}$ is a filter on $\mathfrak{A}$, (2) $I \subset \alpha$, (3) $\alpha^{\cup} \in \mathfrak{U}$, (4) For every α there exists a β such that $\beta\beta \subset \alpha$. It is proved that every u.r.a. is a closure algebra with closure defined by $\overline{R} = \bigcap \alpha R \alpha$. A beginning is made in developing u.r.a. as an algebraization of uniform space theory. (Received January 17, 1952.)

Statistics and Probability

312*t*. H. W. Becker: *Permutation isomorphs.* I.

The number of n letter words in which the alphabetically kth letter may not occur prior to the kth position (factorial words) is $f_n = n!$. An n in the mth position of an n letter permutation corresponds to a terminal m in an n letter factorial word. This isomorphism between f_n and the $P_n = n!$ permutations makes $abc \cdots n = f_n \cap P_n$ self-correspondent. Double entry tables $f_{n,m} \to P_{n,m}$ then furnish many new classifications of permutations, based on the notions of convergent, and omega. The $(n-1)$th, $(n-2)$th, $\cdots$ convergents of a permutation are the permutations remaining on removal of the letters $n, n-1, \cdots$. The letter k is the omega of the kth convergent. Thus ${}_{\mathrm{I}}f_{n,m} = m \cdot {}_{\mathrm{I}}f_{n,m} + (n-m+1) \cdot {}_{\mathrm{I}}f_{n-1,m-1}$, the subset characterized by n letters m different$\to {}_{\mathrm{I}}P_{n,m}$, the n letter permutations whose omegas occupy m different positions in their convergents (enumerating also those with $m-1$ inversions, or m ascending runs, J. Riordan, Proceedings of the American Mathematical Society vol. 2 (1951) p. 429). Likewise ${}_{\mathrm{III}}f_{n,m} = (n-1) \cdot {}_{\mathrm{III}}f_{n-1,m} + {}_{\mathrm{III}}f_{n-1,m-1}$, characterized by m a's$\to {}_{\mathrm{III}}P_{n,m}$, characterized by m convergents with initial omegas (or else, by m cycles, J. Touchard, Acta Math. vol. 70 (1939) p. 243). Of the two interpretations of $n!$: P_n is the most important to physical science; f_n, to aesthetics. They have further isomorphisms, in terms of square and triangular chessboards. (Received January 8, 1952.)

313*t*. H. W. Becker: *Permutation isomorphs*. II.

Pitch is said to be an element of form in Chinese poetry. If we non-repetitively permute the pitch of each member of a rhyme group in all possible ways, we get L_n pitch-rhyme schemes, where (Riordan) $L_{n+1} = (2n+1)L_n - n(n-1)L_{n-1} = S(L+S)^n$ is the Laguerre number whose generating function is $e^{tL} = e^{t/(1-t)}$. If the first member of each rhyme group is exempted from permutation, we get $R_n = n!$ pitch-rhyme schemes, whose generating function is $e^{tR} = e^{-\log(1-t)}$, which also generates the cycles of substitutions $S_n = n!$ (Touchard, ibid.). Some classification isomorphisms $S_{n,m} \to R_{n,m}$ are the well known tables of m cycles$\to m$ different letters, and m unit cycles $\to m$ singletons; and the new tables, of n in mth cycle$\to$terminal m, m the bottom letter of the first cycle$\to$last "a" in mth position, last underanged letter$\to$last singleton, first m letters all in different cycles$\to$initial ascending run of length m, $n-m-1$ cocyclic pairs of consecutive letters$\to$m non-couplets. These may be specialized to $S^{(k)}_{n,m} \to R^{(k)}_{n,m} = (n-1)S^{(k)}_{n-1,m} + (n-1)_{k-1} \cdot S^{(k)}_{n-1,m-1}$, characterized by at least k letters in each cycle, or rhyme group, whose generating function is $\exp tS^{(k)} = \exp\left(\sum_k^\infty t^i/i\right)$. $S^{(2)}_n = (S-1)^n =$ subfactorial n. (Received January 8, 1952).

314. Miriam A. Lipschutz: *Some results on strong laws*.

Chung and Erdös (*On the application of the Borel-Cantelli lemma*, Trans. Amer. Math. Soc. vol. 72 (1952) pp. 179–186) have proved the following theorem. Let $\{E_n\}$ be an infinite sequence of events with $\sum P(E_n) = \infty$. Under a set of auxiliary conditions involving the joint probabilities of pairs of events E_j, we can conclude that $P(E_n \text{ i.o.}) = 1$ providing the following holds: For every positive integer h there exists a constant c and an integral valued function $H(n)$ such that for every $k \geqq H(n)$ and $n \geqq h$ we have (M): $P(E_k | E'_h \cdots E'_n) > cP(E_k)$. On the basis of a general result on the validity of (M) for (a) certain types of events connected with sums of independent r.v., (b) recurrent events, the author has applied this theorem to obtain strong laws for the distribution of the number of positive sums of independent r.v., covering all cases where the arcsine law has been shown to hold. Similarly she has obtained strong upper bounds for the number of realizations of $\mathcal{E}$ in n trials for recurrent events $\mathcal{E}$ with infinite variance when the distribution of the recurrence time is of the form $P(X>x)=h(x)/x^{\alpha}$, with $\lim_{x\to\infty} h(cx)/h(x)=1$ and $\alpha=1/2$ (Feller, *Fluctuation theory of recurrent events*, Trans. Amer. Math. Soc. vol. 67, p. 106). This includes the result of Chung-Hunt on the number of zeros in $\sum \pm 1$ (Ann. of Math. vol. 50, p. 389). (Received January 9, 1952.)

Topology

315. R. D. Anderson: *A monotone interior dimension-raising mapping defined over S^3.*

The author shows that there exists a continuous collection G of mutually exclusive compact continua filling up S^3 such that G with respect to its elements as points is homeomorphic to a space in which the Hilbert cube is imbedded. This result settles affirmatively the question as to whether there exists an interior map of a manifold on a space of higher dimension (S. Eilenberg, *On the problems of topology*, Ann. of Math. vol. 50 (1949) pp. 247–260). As a corollary of the methods used it is also established that if S is any compact triangulable n-manifold ($n>1$) and ϵ is any positive number, there exists a monotone interior mapping f of S onto S such that the inverse image under f of any point S is within ϵ of each point of S. (Received January 10, 1952.)

316*t*. D. O. Ellis: *Orbital topologies.*

Let S be a set and γ a mapping of S into itself. Define: x is an accumulation point of X in S_γ if and only if x is in the γ-orbits of infinitely many distinct points of X. S_γ is a T_1 space but not, in general, a T_2 space. The major results are: (1) *If γ is finite-to-one, then γ is continuous in S_γ*; (2) *If δ is finite-to-one mapping of S into itself and δ commutes with γ, then δ is continuous in S_γ*; (3) *If γ is biuniform and onto, γ is a homeomorphism of S_γ*; and, as a novelty, (4) *If k is any infinite cardinal, there is a dense-in-itself T_1 space of cardinal 2^k whose homeomorphism group contains an Abelian subgroup of cardinal 2^k of involutory homeomorphisms.* (Received December 17, 1951.)

317*t*. S. T. Hu: *A cohomology theory with higher coboundary operators.* III. *The homotopy axiom and the groups for spheres.*

In our first note of this series [Indagationes Mathematicae vol. 11 (1949) pp. 418–424], a generalization of the classical cohomology theory is given by introducing coboundary operators of higher order. In our second note [Indagationes Mathematicae vol. 12 (1950) pp. 1–7], the algebraic axioms, the exactness axiom, and the excision axiom of Eilenberg and Steenrod are verified; and the m-dimensional (p, q)-cohomology groups of a single point are computed. At that time, the author was not able to see if the homotopy axiom is satisfied. In a recent note of J. W. Keesee [Ann. of Math. vol. 54 (1951) pp. 247–249], it is shown that the continuity axiom and the algebraic axioms imply the homotopy axiom. This opens an easy way to check the homotopy axiom for our generalized theory. In the present note, the author indicates the verification of the homotopy axiom for our generalized theory by means of Keesee's theorem. By means of these proved axioms, the m-dimensional (p, q)-cohomology groups of an n-sphere over a coefficient group G are computed in the last section of the present note. (Received January 8, 1952.)

318*t*. S. T. Hu: *On local structure of finite-dimensional groups.*

Let G denote a connected locally compact separable metric topological group of finite dimension n. In a work of Deane Montgomery [Ann. of Math. vol. 52 (1950) pp. 591–605], it is proved that G contains a compact zero-dimensional set Z and an invariant connected, locally connected, locally compact n-dimensional local group C such that $U=ZC$ is an open set containing the identity element e and that the correspondence $(z, c)\rightarrow zc$ defines a homeomorphism of the topological product space $Z\times C$ onto U. In general, it is not known whether or not the set Z can be selected to be a subgroup of G. In the present paper, it is proved by means of local group extensions that the answer is affirmative at least for the particular case that the center of G is locally connected. As a corollary, we have the assertion that if G has no arbitrarily small subgroups, then G must be locally connected. (Received January 8, 1952.)

L. W. Cohen
Associate Secretary