$r = 1, s = 0(0.5)10(1)40(2)100; r = 2, s = 0(1)40(2)100;$

$r = 3, s = 0(1)20(2)68(4)100; r = 4, s = 0(1)14(2)40(4)100;$

$r = 5, s = 0(2)60(4)100; r = 6, s = 0(2)100; r = 7(1)9, s = 0(4)100;$

$r = 10, s = 0(5)100; r = 11(1)15, s = 0(10)100.$

Most values are given to 9 or 10 decimals, though for the smaller values of k a constant number (9 or 10) of significant figures is provided so as to allow the complete solution of (*) to be calculated to a high degree of accuracy. The remaining tables contain joining factors and auxiliary functions, a description of which is beyond the scope of this review.

In conclusion, the reviewer thinks that the computation of the several Mathieu functions themselves would be very welcome. At least for the periodic Mathieu functions proper, this would not be too difficult for table-makers using high-speed electronic computers!

C. J. BOUWKAMP

Die zweidimensionale Laplace-Transformation. Eine Einführung in ihre Anwendung zur Lösung von Randwertproblemen nebst Tabellen von Korrespondenzen. By D. Voelker and G. Doetsch. (Lehrbücher und Monographien aus dem Gebiete der Exakten Wissenschaften, Mathematische Reihe, vol. 12.) Basel, Birkhäuser, 1950. 259 pp. 43 Swiss fr.

The motivation of this book is described by the authors in the preface in the following words. "The classical one-dimensional Laplace transformation now belongs to the common heritage of mathematicians and technologists, and since the publication of the monograph *Theorie und Anwendung der Laplace-Transformation*, books have been written on it in almost all cultured languages. By contrast, the two-dimensional (or double) Laplace transformation has been used only occasionally, in some memoirs. There is no systematic presentation of its theory and applications, or an elucidation of its distinctive features. Such a presentation is offered in this book which consists largely of unpublished material." (Reviewer's translation.)

The emphatic reference to the senior author's well known treatise should not mislead the reader into expecting the same sort of book here. On the one hand, the present book is based on Lebesgue's integration theory while Riemann integrals were used in Doetsch's book; and on the other hand, in contradistinction to the basic character of the earlier work, the orientation of the book under review is towards the applications, especially partial differential equations. This more practical orientation is shown by leaving aside basic questions of purely mathematical interest (for instance the highly complex behaviour of non-absolutely convergent double Laplace integrals). It is also shown by a mellower attitude towards the point of view of the engineer. On page 20 it is pointed out that even if one does not know Lebesgue's theory of integration, one can use with impunity theorems based on that theory as long as one follows the "rules of the game"; and readers of Doetsch's earlier work will be surprised to see that on page 51 the description of the formal process of solving boundary value problems by means of the double Laplace transformation is merely followed by the remark that a *careful* worker (reviewer's emphasis) will then find out under what conditions and in which sense the formal solution actually solves the boundary value problem. The most conspicuous, and the most welcome, sign of this practical orientation is the collection of tables of operations and transform pairs which occupies almost one half of the book.

The relation

$$f(s) = \int_0^\infty e^{-st}F(t)dt \tag{1}$$

is indicated either as $f(s) = \mathfrak{L}\{F(t)\}$ or, more often, as $f(s) \bullet - \circ F(t)$ or $F(t) \circ - \bullet f(s)$. If necessary, the variables involved in the transformation are noted. Thus

$$f(u, y) \overset{u}{\bullet} - \overset{x}{\circ} F(x, y)$$

means

$$f(u, y) = \int_0^\infty e^{-ux}F(x, y)dx.$$

The double Laplace transform

$$f(u, v) = \int_0^\infty \int_0^\infty e^{-ux-vy}F(x, y)dxdy \tag{2}$$

is written as

$$f(u, v) \underset{v}{\overset{u}{\bullet}} = \underset{y}{\overset{x}{\circ}} F(x, y)$$

or simply as $f(u, v) \bullet = \circ F(x, y)$. There are already several symbols for the Laplace transformation and the introduction of a new notation is not in itself desirable, yet in the present case it can be defended on the grounds of its being the only notation which lends itself conveniently to an indication of the number and names of the variables involved in the transformation.

The whole work is organized in two parts. Part I (137 pp.), on the theory and application to boundary value problems of the double Laplace transformation, is by both authors; Part II (107 pp.), containing the tables, is compiled by Voelker.

In Chapter 1 of Part I first the basic properties of the Laplace transform (1) are recalled (§1). Then the double Laplace transform is introduced (in §2) as the absolutely convergent Lebesgue double integral (2). For this, there are associated abscissae of (absolute) convergence, with properties similar to the associated radii of convergence of double power series. By Fubini's theorem, the double Laplace transform is reduced to a succession of two Laplace transforms (§3), and it is shown (in §4) that in case either of the two parameters u, v is fixed, the double Laplace transform is an analytic function of the other, regular in a half-plane which is determined by the associated abscissa of convergence. This leads to the expression of the double Laplace transform of $x^m y^n F(x, y)$ in terms of f. Then follow the so-called rules (§§5–7).

Almost the whole of the remainder of Part I is devoted to partial differential equations. The domain is always the quadrant $x \geqq 0$, $y \geqq 0$ which arises in many problems, and especially naturally with parabolic partial differential equations. From the work of §3 it follows that in principle all the work could be based on the (one-dimensional) Laplace transformation, and some of the problems presented here have previously been so handled; but the authors point out certain advantages of the double Laplace transformation. The general method is as follows. First the double Laplace transformation is applied formally to obtain a tentative solution expressed in terms of the coefficients of the partial differential equation and certain boundary (and initial) values, and then an independent investigation is carried out to show under which conditions the formal result represents an actual solution of the problem.

Chapters 2–5 are concerned with partial differential equations in two independent variables. In Chapter 2, after some general remarks (§8), the partial differential equation

$$F_x + pF_y + qF = \Phi(x, y), \qquad x \geqq 0,\ y \geqq 0,$$

is discussed (§9) where p and q are constants, and $p \neq 0$. With the boundary values

$$F(x, 0) = A(x) \circ\!-\!\bullet\ a(u), \qquad F(0, y) = B(y) \circ\!-\!\bullet\ b(v)$$

the double Laplace transform of the solution is

$$f(u, v) = \frac{\phi(u, v) + pa(u) + b(v)}{u + pv + q}. \tag{3}$$

From here on two cases must be distinguished. If $p>0$, the denominator in (3) is always positive for sufficiently large (positive) u and v, and the solution can be obtained, without any quantitative restrictions on the data, by means of the rules. On the other hand, if $p<0$ the condition

$$\phi(-pv - q, v) + pa(-pv - q) + b(v) = 0 \tag{4}$$

must be satisfied identically in v to make (3) regular for all sufficiently large u, v, and this condition imposes a quantitative restriction on the data. The situation is further elucidated by reference to the theory of characteristics. As an application of the double Laplace transformation the work is not very impressive, for the solution can be obtained by elementary methods as easily as, if not more easily than, by the double Laplace transformation. Yet the example is well chosen in that it aptly illustrates the general method, and also shows that in the application of the double Laplace transformation to boundary value problems one needs general operational rules, such as the interpretation of (4), even more than a large number of particular transform pairs.

In Chapter 3, §10 contains a similar discussion of the parabolic partial differential equation

$$F_{xx} - F_y + \Phi(x, y) = 0.$$

At first three boundary values,

$$F(0, y) \circ\!-\!\bullet\, b(v), \qquad F_x(0, y) \circ\!-\!\bullet\, b_1(v), \qquad F(x, 0) \circ\!-\!\bullet\, a(u)$$

enter, and

$$f(u, v) = \frac{-a(u) + ub(v) + b_1(v) - \phi(u, v)}{u^2 - v}.$$

The regularity of f demands

$$-a(v^{1/2}) + v^{1/2}b(v) + b_1(v) - \phi(v^{1/2}, v) = 0 \tag{5}$$

and thus sets up a compatibility condition. Various boundary value problems are discussed. Other partial differential equations in this chapter are: the wave equation (§11), $F_{xy}+pF+\Phi(x, y)=0$ (§12), the telegraphist's equation (§13), and Poisson's equation (§14). In all these cases the authors discuss those boundary value problems which can be handled most conveniently by the operational calculus, rather than those which arise most naturally in physical problems. As is well known, the discrepancy is greatest in the case of elliptic equations.

The general linear partial differential equation of order two (in two independent variables) with constant coefficients is discussed in Chapter 4. §15 correlates the double Laplace transform method with Green's formula, §16 deduces compatibility conditions (whose number depends on the coefficient matrix of the equation). In the remaining paragraphs (§§17–20) the discussion of hyperbolic and parabolic equations is taken up in more detail.

In Chapter 5 (§§21–24) the discussion of the system of partial differential equations with constant coefficients

$$F_x + p_1F_y + q_{11}F + q_{12}G = \Phi(x, y),$$
$$G_x + p_2G_y + q_{21}F + q_{22}G = \Psi(x, y)$$

is taken up. The number of compatibility conditions is 0, 1, 2 according as $0 \leqq p_1 \leqq p_2$, $p_1 < 0 \leqq p_2$, $p_1 \leqq p_2 < 0$.

Chapter 6 is on partial differential equations in three independent variables. After a general introduction (§25), the partial differential equation

$$F_{xx} - F_y + F_{zz} = 0, \qquad x \geqq 0,\ y \geqq 0,\ 0 \leqq z \leqq l, \tag{6}$$

is discussed (§26). The double Laplace transformation is applied with respect to x and y, and (6) is transformed into an ordinary differential equation which can be solved explicitly. The inverse transformation is considerably more difficult in this case. Three methods of it are illustrated in §§27–29.

Part I concludes with Chapter 7 (§§30, 31) on functional relations involving Bessel functions and Laguerre polynomials. A brief list of references is added.

Although Part I is unlikely to become as basic as the senior author's well known treatise on the Laplace integral, in its more restricted sphere it should prove useful.

In the preface the authors explain that one-dimensional operational calculus was for a long time handicapped by the absence of an adequate "dictionary." In order to forestall a similar situation with regard to the double Laplace transformation, they provide extensive tables in Part II, by Voelker.

Table A contains the "grammar" or operational rules. 1. Fundamental operations (64 entries). Beside the formulas for double Laplace transforms of partial derivatives (general formulas, and explicit relations up to order 4), there are other useful formulas of which a sample is

$$(x+y)^{-1}F(x, y) \circ = \bullet \int_u^\infty f(\lambda, v+\lambda-u)d\lambda.$$

2. Double Laplace transforms obtained from Laplace transforms (86 entries). Sample:

$$v^{-1}\phi(u+v^{1/2}) \bullet = \bullet\ \Phi(x) \operatorname{erfc} \frac{x}{2y^{1/2}} \quad \text{where} \quad \phi(u) \bullet - \bullet\ \Phi(x).$$

3. Difference quotients of Laplace transforms (63 entries). Samples:

$$\frac{\phi(u)-\phi(v)}{u-v} \bullet = \circ - \Phi(x+y) \text{ where } \phi(u) \bullet - \circ\ \Phi(x),$$

$$\frac{\phi(u)-\phi((v^2+\alpha^2)^{1/2})}{(v^2+\alpha^2)^{1/2}(u-(v^2+\alpha^2)^{1/2})} \bullet = \circ - \int_0^y J_0(\alpha(y^2-\xi^2)^{1/2})\Phi(x+\xi)d\xi.$$

4. More involved operations (134 entries). Sample:

$$\frac{1}{u} f\left(u+\frac{\alpha}{v}, v\right) \bullet = \circ \int_0^x J_0(2(\alpha\xi(x-\xi))^{1/2})F(\xi, y)d\xi.$$

5. Difference quotients of double Laplace transforms (41 entries). Sample:

$$\frac{f(u, v)-f(v, v)}{u-v} \bullet = \circ - \int_0^v F(x+\eta, y-\eta)d\eta.$$

6. Laplace transforms derived from double Laplace transforms (10 entries). Sample:

$$f(u, 0) \overset{u}{\bullet} - \overset{x}{\circ} \int_0^\infty F(x, \eta)d\eta.$$

Table B contains the "dictionary" or transform pairs. These are arranged according to the $f(u, v)$. 1. Rational functions (177 entries). 2. Irrational algebraic functions and powers with arbitrary index (97 entries). 3. Logarithms (27 entries). 4. Exponential functions (28 entries). 5. Hyperbolic functions (13 entries). 6. The exponential integral function and similar functions (11 entries). 7. Confluent hypergeometric functions (13 entries). 8. Other functions (10 entries).

Table C contains a list of notations for special functions (110 entries).

These very detailed tables are likely to become an indispensable tool for the practical application of the double Laplace transformation.

The publishers must be congratulated on the excellent performance of a typographical job (the tables) which presents considerable technical and financial difficulties.

A. ERDÉLYI

Mathematische Grundlagenforschung. By Arnold Schmidt. (Enzyklopädie der Mathematischen Wissenschaften, Band I_1, Heft 1, Teil II.) Leipzig, 1950. 48 pp.

This article was originally written in 1939, but was revised in 1948–49, so as to include reference to more recent contributions. It consists of an exposition of "those parts of foundation studies which are either directly concerned with the construction of mathematics, or in which the application of mathematical methods has proved fruitful." The author confines himself almost entirely to the foundations of arithmetic, and does not attempt to deal with such topics as set theory, group theory, or geometry, nor with the problems proper to mathematical logic itself.

The article consists of four parts: (A) *Axiomatik und allgemeine Beweistheorie*; (B) *Kodifikation und Beweistheorie der Zahlenlehr*; (C) *Die logische Begrundung der Mathematik*; and (D) *Intuitionistische Mathematik.*

In Part (A), the author explains what is meant by the formalization of a mathematical system, and introduces some metamathematical terms. He then shows (following Gödel) that every system which contains arithmetic also contains its own syntax in arithmetical form, and proceeds to sketch a proof of Gödel's theorem (as strengthened by Rosser): that a system which contains its own syntax in arithmetical form cannot be both consistent and complete. Among other results of a negative character mentioned here are a second theorem of Gödel, that a consistent system containing its own syntax in arithmetical form cannot be shown to be consistent by any proof which can be formalized within the system, and the theorem of Tarski that, in a system which contains its own syntax in arithmetical form, one cannot define truth for the system itself.

Part (B) begins with a brief sketch of the theory of recursive functions. The result of Péter is cited, that one can keep on getting new recursive functions by increasing the number of variables in primitive recursions. Some of the various equivalent methods of defining gen-