REAL ROOTS OF DIRICHLET L-SERIES

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Let k be a positive integer. Let χ be a real, non-principal character (mod k) and

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

be the corresponding L-series, which converges uniformly for $R(s) \geqq \epsilon > 0$. If it could be shown that uniformly in k there is no real zero of $L(s, \chi)$ for

$$s \geqq 1 - \frac{A}{\log k},$$

where A is a constant, then the existing theorems on the distribution of primes in arithmetic progressions could be greatly improved (see [1]).[1] Moreover by Hecke's Theorem (see [2]) it would follow that uniformly in k

$$L(1, \chi) > \frac{B}{\log k}$$

where B is a constant. This would be a considerable improvement over Siegel's Theorem (see [3]), and would lead to an improved lower bound for the class number of an imaginary quadratic field.

In the present paper, we shall show that for $2 \leqq k \leqq 67$, $L(s, \chi)$ has no positive real zeros. By combining this information with the results of [1], we infer very sharp estimates on the distribution of primes in arithmetic progressions of difference k for $k \leqq 67$.

The methods used for $k \leqq 67$ certainly will suffice for many other k's greater than 67. They may possibly suffice for all k, but we can find no proof of this.[2]

In [5], S. Chowla has considered the positive real zeros of $L(s, \chi)$, and shown that for many explicit k's, no positive real zeros exist. However Chowla could not settle whether his methods would suffice

Presented to the Society, September 1, 1949; received by the editors August 30, 1948.

[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] These methods have been tried on all $k \leqq 227$ and it has been ascertained that except for the cases $k=148$ and $k=163$, $L(s, \chi)$ has no positive real zeros for $2 \leqq k \leqq 227$. Cases $k=148$ and $k=163$ are now being studied and any results obtained about them will appear in the Journal of Research of the National Bureau of Standards.

to handle the difficult cases $k=43$ and $k=67$. In [6], Heilbronn has shown that there exist values of k for which Chowla's methods are certainly inadequate.

THEOREM 1. *If* χ *is non-principal* (mod k) *and* $\chi(-1)=1$, *then for all* s

$$L(s, \chi) = \sum_{\alpha=1}^{\infty} \frac{2s(s+1)\cdots(s+2\alpha-1)}{4^{\alpha}(2\alpha)!k^{s+2\alpha}}(2^{s+2\alpha}-1)\zeta(s+2\alpha) \cdot \sum_{n=1}^{[k/2]} \chi(n)(k-2n)^{2\alpha}.$$

PROOF. For $s>1$, we have

$$\begin{aligned}
L(s, \chi) &= 2^s \sum_{N=0}^{\infty}\sum_{n=1}^{k-1} \frac{\chi(n)}{(2kN+2n)^s} \\
&= 2^s \sum_{N=0}^{\infty} \frac{1}{k^s(2N+1)^s} \sum_{n=1}^{k-1} \chi(n)\left(1 - \frac{k-2n}{k(2N+1)}\right)^{-s} \\
&= 2^s \sum_{N=0}^{\infty} \frac{1}{k^s(2N+1)^s} \sum_{n=1}^{k-1} \chi(n)\left\{1 + s\frac{k-2n}{k(2N+1)}\right. \\
&\quad + \frac{s(s+1)}{2!}\left(\frac{k-2n}{k(2N+1)}\right)^2 \\
&\quad \left. + \frac{s(s+1)(s+2)}{3!}\left(\frac{k-2n}{k(2N+1)}\right)^3 + \cdots\right\} \\
&= 2^s \sum_{N=0}^{\infty} \frac{1}{k^s(2N+1)^s}\left\{\frac{s}{k(2N+1)}\sum_{n=1}^{k-1}\chi(n)(k-2n)\right. \\
&\quad \left. + \frac{s(s+1)}{2!k^2(2N+1)^2}\sum_{n=1}^{k-1}\chi(n)(k-2n)^2 + \cdots\right\} \\
&= \frac{s}{2k^{s+1}}\left(\sum_{N=0}^{\infty}\left(\frac{2}{2N+1}\right)^{s+1}\right)\sum_{n=1}^{k-1}\chi(n)(k-2n) \\
&\quad + \frac{s(s+1)}{4(2!)k^{s+2}}\left(\sum_{N=0}^{\infty}\left(\frac{2}{2N+1}\right)^{s+2}\right)\sum_{n=1}^{k-1}\chi(n)(k-2n)^2 + \cdots \\
&= \frac{s}{2k^{s+1}}(2^{s+1}-1)\zeta(s+1)\sum_{n=1}^{k-1}\chi(n)(k-2n) \\
&\quad + \frac{s(s+1)}{4(2!)k^{s+2}}(2^{s+2}-1)\zeta(s+2)\sum_{n=1}^{k-1}\chi(n)(k-2n)^2 + \cdots .
\end{aligned}$$

Since χ is non-principal, we have $k>2$, and so if k is even, we have $\chi([k/2])=\chi(k/2)=0$. Now since $\chi(-1)=1$,

$$\begin{aligned}\sum_{n=1}^{k-1}\chi(n)(k-2n)^{2\alpha}&\\ &=\sum_{n=1}^{[k/2]}\chi(n)(k-2n)^{2\alpha}+\sum_{n=[k/2]+1}^{k-1}\chi(n)(2n-k)^{2\alpha}\\ &=\sum_{n=1}^{[k/2]}\chi(n)(k-2n)^{2\alpha}+\sum_{n=k-[k/2]}^{k-1}\chi(n)(k-2(k-n))^{2\alpha}\\ &=\sum_{n=1}^{[k/2]}\chi(n)(k-2n)^{2\alpha}+\sum_{n=1}^{[k/2]}\chi(k-n)(k-2n)^{2\alpha}\\ &=2\sum_{n=1}^{[k/2]}\chi(n)(k-2n)^{2\alpha}.\end{aligned}$$

Similarly, we prove $\sum_{n=1}^{k-1}\chi(n)(k-2n)^{2\alpha+1}=0$.

Thus we infer that the equation stated is valid for $s>1$.

Now since

$$\left|\sum_{n=1}^{[k/2]}\chi(n)(k-2n)^{2\alpha}\right|\leqq\frac{k}{2}(k-2)^{2\alpha},$$

we see that the series on the right converges absolutely and uniformly for all s, and so our theorem follows by analytic continuation.

THEOREM 2. *If χ is non-principal* (mod k) *and* $\chi(-1)=-1$, *then for all* s

$$L(s,\chi)=\sum_{\alpha=0}^{\infty}\frac{s(s+1)\cdots(s+2\alpha)}{4^{\alpha}(2\alpha+1)!k^{s+2\alpha+1}}(2^{s+2\alpha+1}-1)\zeta(s+2\alpha+1)\cdot\sum_{n=1}^{[k/2]}\chi(n)(k-2n)^{2\alpha+1}.$$

The proof is similar to the proof of Theorem 1.

Although these theorems hold for any non-principal χ, we shall use them only for real non-principal χ. We assume henceforth that χ is real and non-principal. We let Σ_M denote

$$\sum_{n=1}^{[k/2]}\chi(n)(k-2n)^{M}.$$

For sufficiently large M (certainly for $M\geqq k$), the initial term

$$\chi(1)(k-2)^{M}$$

of Σ_M dominates the remaining terms, and we infer that $\Sigma_M>0$. If by good chance $\Sigma_M \geqq 0$ for all $M\geqq 1$, then by Theorem 1 or Theorem 2 we infer that $L(s, \chi)>0$ for $s>0$, and hence that $L(s, \chi)$ has no positive real zeros. For $k\leqq 67$, this happens in a majority of cases.

When considering positive real zeros of $L(s, \chi)$ it suffices to restrict attention to primitive χ's (and to the k's for which there are primitive χ's. See [4, §125]). For primitive χ's, $\Sigma_M\geqq 0$ for $M\geqq 1$ for each $k\leqq 67$ except 43 and 67. Moreover for each such k, the proof of $\Sigma_M\geqq 0$ is easily accomplished by grouping the terms in groups, each of which is non-negative. Typical such groups are:

I. A^M-B^M, where $A>B$.

II. $A^M-B^M-C^M$, where $A\geqq B+C$.

III. $A^M-B^M-C^M+D^M$, where $A+D\geqq B+C$.

For $k=53$, there occurs the group $51^M-49^M-47^M+45^M-43^M+41^M+39^M-37^M$, which we show to be non-negative by writing it as $(44+7)^M-(44+5)^M-(44+3)^M+(44+1)^M-(44-1)^M+(44-3)^M+(44-5)^M-(44-7)^M$, and expanding each term by the binomial theorem.

For $k=43$ or 67, we have $\Sigma_3<0$, so that the series in Theorem 2 does not consist entirely of non-negative terms. However, we can show that the initial positive term outweighs the negative terms. We give the proof for $k=67$, since the proof for $k=43$ is similar and easier.

By the functional equation for $L(s, \chi)$ (see [4, §128]) it follows that if $L(s, \chi)$ has a zero ρ with $1/2<\rho<1$, then it has a zero ρ with $0<\rho<1/2$. As it is known that $L(s, \chi)>0$ for $1\leqq s$, it suffices to prove $L(s, \chi)>0$ for $0\leqq s\leqq 1/2$. So we take $k=67$ and $0\leqq s\leqq 1/2$. By Theorem 2,

$$L(s, \chi) = \frac{2^{s+1}-1}{67^s}\left\{\frac{s\zeta(s+1)}{67}\Sigma_1 + \frac{s(s+1)(s+2)}{3!(67)^3}\,\frac{2^{s+3}-1}{4(2^{s+1}-1)}\zeta(s+3)\Sigma_3+\cdots\right\},$$

where now $\Sigma_M=\sum_{n=1}^{33}\chi(n)(67-2n)^M$. For $s>0$,

$$\begin{aligned}\zeta(s+1)-\frac{1}{s} &= \sum_{n=1}^{\infty}\frac{1}{n^{s+1}}-\int_1^{\infty}\frac{dx}{x^{s+1}}\\ &= \sum_{n=1}^{\infty}\left\{\frac{1}{n^{s+1}}-\int_n^{n+1}\frac{dx}{x^{s+1}}\right\}\\ &> 0.\end{aligned}$$

So for $s\geqq 0$, $s\zeta(s+1)\geqq 1$. Also $\Sigma_1=67$. So

$$\frac{s\zeta(s+1)}{67}\Sigma_1 \geqq 1. \tag{1}$$

For $0 \leqq s$

$$s\,\frac{2^{s+2\alpha+1}-1}{4^\alpha(2^{s+1}-1)} < \frac{2s}{2-2^{-s}} \quad \text{and} \quad \frac{d}{ds}\left(\frac{2s}{2-2^{-s}}\right) > 0.$$

So for $0 \leqq s \leqq 1/2$

$$s\,\frac{2^{s+2\alpha+1}-1}{4^\alpha(2^{s+1}-1)} \leqq \frac{2(1/2)}{2-2^{-1/2}} < 0.77346.$$

Also

$$\frac{(s+1)(s+2)}{3!} \leqq \frac{(3/2)\cdot(5/2)}{3!} = \frac{5}{8}\,\cdot$$

Since $\Sigma_3 = -102{,}845$, we infer

$$\begin{aligned}
&\frac{s(s+1)(s+2)}{3!(67)^3}\,\frac{2^{s+3}-1}{4(2^{s+1}-1)}\,\zeta(s+3)\,\Sigma_3 \\
&\qquad \geqq -\frac{5}{8}\,\frac{1}{(67)^3}\,(0.77346)\zeta(3)(102{,}845) \\
&\qquad \geqq -\frac{5}{8}\,(0.77346)(1.20206)\,\frac{102{,}845}{300{,}763} \\
&\qquad > -0.199.
\end{aligned} \tag{2}$$

Now for $M \geqq 1$,

$$\begin{aligned}
\Sigma_M = {} & \{(57+8)^M - (57+6)^M - (57+4)^M + (57+2)^M - 57^M \\
& + (57-2)^M - (57-4)^M - (57-6)^M + (57-8)^M\} \\
& + \{(43+4)^M - (43+2)^M - 43^M - (43-2)^M + (43-4)^M\} \\
& + 37^M + 35^M + \cdots \\
> {} & -57^M + \frac{M(M-1)}{2!}\,57^{M-2}\{2\cdot 8^2 - 2\cdot 6^2 - 2\cdot 4^2 + 2\cdot 2^2\} \\
& + \frac{M(M-1)(M-2)(M-3)}{4!}\,57^{M-4}\{2\cdot 8^4 - 2\cdot 6^4 \\
& - 2\cdot 4^4 + 2\cdot 2^4\} + \cdots \\
& - 43^M + \frac{M(M-1)}{2!}\,43^{M-2}\{2\cdot 4^2 - 2\cdot 2^2\} + \cdots
\end{aligned}$$

$$\geqq -57^M\left(1-\frac{16M(M-1)}{57^2}\right)-43^M\left(1-\frac{12M(M-1)}{43^2}\right).$$

In particular, if $\alpha \geqq 2$, then

$$\begin{aligned}\Sigma_{2\alpha+1} &\geqq -57^{2\alpha+1}\left(1-\frac{16(2\alpha+1)2\alpha}{57^2}\right)\\ &\quad -43^{2\alpha+1}\left(1-\frac{12(2\alpha+1)2\alpha}{43^2}\right)\\ &\geqq -57^{2\alpha+1}\left(1-\frac{320}{3249}\right)-43^{2\alpha+1}\left(1-\frac{240}{1849}\right)\\ &\geqq -57^{2\alpha+1}\frac{2929}{3249}-43^{2\alpha+1}\frac{1609}{1849}.\end{aligned}$$

So for $0 \leqq s \leqq 1/2$,

$$\begin{aligned}&\sum_{\alpha=2}^{\infty}\frac{s(s+1)\cdots(s+2\alpha)}{(2\alpha+1)!(67)^{2\alpha+1}}\frac{2^{s+2\alpha+1}-1}{4^\alpha(2^{s+1}-1)}\zeta(s+2\alpha+1)\Sigma_{2\alpha+1}\\ &\quad\geqq -\sum_{\alpha=2}^{\infty}\frac{s(s+1)\cdots(s+2\alpha)}{(2\alpha+1)!(67)^{2\alpha+1}}\frac{2^{s+2\alpha+1}-1}{4^\alpha(2^{s+1}-1)}\zeta(s+2\alpha+1)\\ &\quad\quad\cdot\left\{57^{2\alpha+1}\frac{2929}{3249}+43^{2\alpha+1}\frac{1609}{1849}\right\}\\ &\quad\geqq -\sum_{\alpha=2}^{\infty}\frac{s(s+1)\cdots(s+4)}{5!(67)^{2\alpha+1}}\frac{2^{s+2\alpha+1}-1}{4^\alpha(2^{s+1}-1)}\zeta(5)\\ &\quad\quad\cdot\left\{57^{2\alpha+1}\frac{2929}{3249}+43^{2\alpha+1}\frac{1609}{1849}\right\}\\ &\quad\geqq -\frac{63}{128}(0.77346)(1.03693)\sum_{\alpha=2}^{\infty}\left\{\left(\frac{57}{67}\right)^{2\alpha+1}\frac{2929}{3249}\right.\\ &\quad\quad\left.+\left(\frac{43}{67}\right)^{2\alpha+1}\frac{1609}{1849}\right\}\\ &\quad\geqq -\frac{63}{128}(0.77346)(1.03693)\left\{\left(\frac{57}{67}\right)^5\frac{4489}{1240}\frac{2929}{3249}\right.\\ &\quad\quad\left.+\left(\frac{43}{67}\right)^5\frac{4489}{2640}\frac{1609}{1849}\right\}\\ &\quad> -0.638.\end{aligned}\tag{3}$$

By (1), (2), and (3), for $0 \leqq s \leqq 1/2$,

$$L(s, \chi) \geqq \frac{2^{s+1} - 1}{67^s} \{1.000 - 0.199 - 0.638\} \geqq \frac{0.163}{(67)^{1/2}} \geqq 0.0199.$$

So $L(s, \chi) > 0$ for $0 \leqq s$.

When $\chi(-1) = -1$, Theorem 2 opens up further interesting possibilities. When $s \to 0$, the first term of the series is bounded away from zero, while the remaining terms approach zero. Thus one can always infer $L(s, \chi) > 0$ for $0 \leqq s \leqq \epsilon$, where ϵ depends on k. Even for ϵ as small as $A/\log k$, this would be a very worthwhile result, as remarked at the beginning of the paper.

For another possibility, let $s = 0$ and -2 in Theorem 2, and evaluate $L(0, \chi)$ and $L(-2, \chi)$ by the functional equation. We infer the known result

$$L(1, \chi) = \frac{\pi}{k^{3/2}} \Sigma_1 \tag{4}$$

and the result

$$L(3, \chi) = \frac{\pi^3}{6k^{7/2}} \{k^2 \Sigma_1 - \Sigma_3\}. \tag{5}$$

From these follow

$$\Sigma_3 = k^{7/2} \left\{ \frac{L(1, \chi)}{\pi} - \frac{6L(3, \chi)}{\pi^3} \right\}. \tag{6}$$

This gives

$$\Sigma_3 \geqq - k^{7/2} \frac{6L(3, \chi)}{\pi^3}.$$

If one could prove independently any appreciably better result, one could derive a sensational inequality for $L(1, \chi)$. For instance, if one could prove

$$\Sigma_3 \geqq - k^{7/2} \frac{4}{\pi^3} \geqq - k^{7/2} \frac{5L(3, \chi)}{\pi^3},$$

one could get by (6)

$$L(1, \chi) > \frac{L(3, \chi)}{\pi^2}.$$

Another possibility is that one can perhaps derive some connection between Σ_1 and Σ_3. For instance, if one could prove

$$\Sigma_3 \geqq -k^2 \log k \, \Sigma_1,$$

then by (4) and (6), we could infer

$$L(1, \chi) > \frac{6L(3, \chi)}{\pi^2(1 + \log k)}.$$

Even this would be a very worthwhile result, since the best known at present is, by Siegel's Theorem,

$$L(1, \chi) > \frac{L(3, \chi)}{k^\epsilon}$$

for $\epsilon > 0$ and large k.

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