CONVERGENCE OF CONTINUED FRACTIONS IN PARABOLIC DOMAINS

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1. **Introduction.** The principal object of this paper is to establish the following theorem.

THEOREM A. *Let $c_1, c_2, c_3, \cdots$ be a sequence of complex numbers such that, for $p=1, 2, 3, \cdots$,*

$$|c_p| - R(c_p e^{i(\phi_p+\phi_{p+1})}) \leqq 2r \cos\phi_p \cos\phi_{p+1}(1-g_{p-1})g_p, \tag{1.1}$$

where $r, \phi_1, \phi_2, \phi_3, \cdots, g_0, g_1, g_2, \cdots$ are real numbers satisfying the inequalities

$$\begin{aligned} &0 < r < 1, \quad -\pi/2 + c \leqq \phi_p \leqq +\pi/2 - c \qquad (0 < c < \pi/2), \\ &\qquad\qquad 0 \leqq g_{p-1} \leqq 1, \qquad\qquad p = 1, 2, 3, \cdots, \end{aligned} \tag{1.2}$$

c and r being independent of p. The continued fraction

$$\cfrac{1}{1+\cfrac{c_1}{1+\cfrac{c_2}{1+\cdots}}} = \mathop{K}_{p=1}^{\infty} \frac{c_{p-1}}{1} \qquad (c_0 = 1) \tag{1.3}$$

converges if, and only if, (a) *some c_p vanishes, or* (b) $c_p \neq 0$, *$p=1, 2, 3, \cdots$, and the series $\sum|d_p|$ diverges, where*

$$d_1 = 1, \quad d_{p+1} = \frac{1}{c_p d_p}, \qquad p = 1, 2, 3, \cdots. \tag{1.4}$$

We note the following particular cases of Theorem A.

(a) The continued fraction

$$\mathop{K}_{p=1}^{\infty} \frac{1}{k_p e^{i\phi_p}} = \frac{1}{k_1 e^{i\phi_1}} \mathop{K}_{p=1}^{\infty} \frac{c_{p-1}}{1}, \quad c_0 = 1, \quad c_p = \frac{e^{-i(\phi_p+\phi_{p+1})}}{k_p k_{p+1}},$$

in which $k_p > 0$, $-\pi/2 + c \leqq \phi_p \leqq +\pi/2 - c$, $0 < c < \pi/2$, converges if, and only if, the series $\sum k_p$ diverges (Stieltjes [6] ($\phi_p = \phi$); E. B. Van Vleck [8]).[1] For an extension of this theorem in a direction different from Theorem A, see Scott and Wall [5].

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

(b) If

$$|c_p| - R(c_p e^{2i\phi}) \leqq 2^{-1} r \cos^2 \phi, \qquad p = 1, 2, 3, \cdots,$$

where $0<r<1$, $-\pi/2<\phi<+\pi/2$, then the continued fraction (1.3) converges if, and only if, (a) some c_p vanishes, or (b) $c_p \neq 0$, $p=1, 2, 3, \cdots$, and the series $\sum|d_p|$, defined by (1.4), diverges (Paydon and Wall [3]). The case $\phi=0$ of this theorem holds with $r=1$ (Scott and Wall [4]).

(c) Inasmuch as

$$\frac{1}{(|c_p|)^{1/2}} \leqq \frac{|d_p| + |d_{p+1}|}{2},$$

it follows from Theorem A that a sufficient condition for convergence of the continued fraction (1.3), satisfying (1.1) and (1.2), is the divergence of the series $\sum(1/(|c_p|)^{1/2})$ (Wall and Wetzel [7]). This sufficient condition is not necessary, as is shown by the example $d_{2p-1}=1$, $d_{2p}=s^p$, $0<s<1$.

2. **Preliminary theorem.** Let $x_p = X_p(z)$ and $x_p = Y_p(z)$ be the solutions of the system of equations

$$(2.1) \qquad -a_{p-1}x_{p-1} + (b_p + z_p)x_p - a_p x_{p+1} = 0, \qquad p = 1, 2, 3, \cdots,$$

under the initial conditions $x_0 = -1$, $x_1 = 0$ and $x_0 = 0$, $x_1 = 1$, respectively. We suppose that $a_0 = 1$, $a_1, a_2, a_3, \cdots$ are constants not zero, $b_1, b_2, b_3, \cdots$ are constants, and $z_1, z_2, z_3, \cdots$ are parameters. The *theorem of invariability* [2, 5] states that if the series

$$(2.2) \qquad \sum |X_p(z)|^2, \qquad \sum |Y_p(z)|^2$$

converge for $z_p = h_p$, $p=1, 2, 3, \cdots$, then they converge uniformly for $|z_p - h_p| \leqq M$, for every finite constant M independent of p. The *determinate case* is said to hold for the continued fraction

$$(2.3) \qquad -\overset{\infty}{\underset{p=1}{K}} \frac{-a_{p-1}}{b_p + z_p}$$

if at least one of the series (2.2) diverges for $z_p = 0$, $p=1, 2, 3, \cdots$. In the contrary event, the *indeterminate case* is said to hold.

THEOREM 2.1. *If* $|b_p| \leqq M$, $p=1, 2, 3, \cdots$, *where* M *is a finite constant independent of* p, *then the determinate case holds for the continued fraction* (2.3) *if, and only if, the series* $\sum|d_p'|$ *diverges, where*

$$(2.4) \qquad d_1' = 1, \qquad d_{p+1}' = \frac{1}{a_p^2 d_p'}, \qquad p = 1, 2, 3, \cdots.$$

PROOF. From the condition imposed upon b_p, and the theorem of invariability, it follows immediately that the determinate case holds if, and only if, at least one of the series (2.2) diverges for $z_p = -b_p$, $p=1, 2, 3, \cdots$. On putting these values of the z_p in (2.1) we find that $|X_{2p}(z)|^2$, $|Y_{2p}(z)|^2$, $|X_{2p+1}(z)|^2$ and $|Y_{2p+1}(z)|^2$ take on the values $|d'_{2p}|$, 0, 0, and $|d'_{2p+1}|$, respectively. Therefore, the determinate case holds if, and only if, the series $\sum |d'_p|$ is divergent.

It is easy to see that when we drop the condition that the $|b_p|$ be bounded, then the determinate case may hold when the series $\sum |d'_p|$ converges. It seems likely, however, that the divergence of the series $\sum |d'_p|$ implies the determinate case whether or not the $|b_p|$ are bounded.

3. **Proof of Theorem A.** Let $\delta > 0$ be chosen sufficiently small in order that

$$r\left[1 + \delta \sec\left(\frac{\pi}{2} - c\right)\right]^2 \leqq 1.$$

Determine numbers a_p^2 by means of the equations

$$c_p = \frac{a_p e^{-i(\phi_p + \phi_{p+1})}}{(1 + \delta \sec \phi_p)(1 + \delta \sec \phi_{p+1})}, \qquad p = 1, 2, 3, \cdots.$$

Let the partial numerators a_p^2 in (2.3) have these values, and there take

$$z_p = i\delta, \qquad b_p + z_p = ie^{i\phi_p}(1 + \delta \sec \phi_p).$$

Then that continued fraction and (1.3) are equivalent, except for an unessential factor. Moreover, by (1.1),

$$|a_p^2| - R(a_p^2) \leqq 2\beta_p\beta_{p+1}(1 - g_{p-1})g_p, \qquad p = 1, 2, 3, \cdots,$$

where $\beta_p = I(b_p) = \cos \phi_p > 0$. Thus, the continued fraction (2.3) is *positive definite* [7, 1]. Since $I(z_p) = \delta > 0$, it follows that the continued fraction (2.3) converges if (a) some a_p vanishes, that is, some c_p vanishes, or (b) $a_p \neq 0$, $p=1, 2, 3, \cdots$, and the determinate case holds. Since the $|b_p|$ are bounded, it follows from Theorem 2.1 that the determinate case holds if the series $\sum |d'_p|$ defined by (2.4) diverges. We note that this series diverges if, and only if, the series $\sum |d_p|$ defined by (1.4) diverges. Therefore, the continued fraction (1.3) converges if (a) some c_p vanishes, or (b) $c_p \neq 0$, $p=1, 2, 3, \cdots$, and the series $\sum |d_p|$, defined by (1.4), diverges, If, on the other hand, this series converges, then the continued fraction diverges by virtue of a theorem of von Koch.

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REMARKS ON THE NOTION OF RECURRENCE

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We give in several lines a simple proof of Poincaré's recurrence theorem.

THEOREM. *Let Ω be a point set of finite Lebesgue measure, and T a one-to-one measure-preserving transformation of Ω into itself.*[1] *Let $B \subset A \subset \Omega$ be measurable sets such that, if $b \in B$, $T^n b \notin A$ for all positive integral n. Then the measure $m(B)$ of B is* 0.

PROOF. First we show that, if $i < j$, $(T^i B)(T^j B) = 0$. Suppose $c \in T^j B$; then from the hypothesis on B it follows that j is the smallest integer such that $T^{-j} c \in A$. Hence $c \notin T^i B$. Now if $m(B) = \delta > 0$, Ω would contain infinitely many disjunct sets $T^n B$, each of measure δ. This contradiction proves the theorem.

The following generalization of the above theorem is trivially obvious: The result holds if we replace the hypothesis that T is measure-preserving by the following: If $m(D) > 0$, $\limsup_i m\{T^i(D)\} > 0$.

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[1] For a discussion in probability language see M. Kac, *On the notion of recurrence in discrete stochastic processes*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 1002–1010.