A PROPERTY OF POWER SERIES WITH POSITIVE COEFFICIENTS

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The following theorem is suggested by a problem in the theory of probability.[1]

Let p_k *be a sequence of non-negative numbers for which* $\sum_0^\infty p_k = 1$, *and let* $m = \sum_1^\infty k p_k \leqq \infty$. *Suppose further that*

$$P(x) = \sum_0^\infty p_k x^k$$

is not a power series in x^t *for any integer* $t > 1$. *Then* $1 - P(x)$ *has no zeros in the circle* $|x| < 1$, *and the series*

$$U(x) = \frac{1}{1 - P(x)} = \sum_0^\infty u_k x^k$$

has the property

$$\lim_{n \to \infty} u_n = 1/m.$$

(*If* $m = \infty$, *we define* $1/m$ *to be zero.*)

We shall first give a proof in case $m < \infty$. The method used is not elementary, but yields somewhat more information than stated in the theorem. Later in this paper an elementary proof is given, valid for both $m < \infty$ and $m = \infty$.

We suppose that $m < \infty$. Let

$$(1) \qquad r_n = \sum_{k=n+1}^\infty p_k, \qquad R(x) = \sum_0^\infty r_n x^n.$$

Then $m = \sum_0^\infty r_k$ and

$$(2) \qquad 1 - P(x) = (1 - x) R(x).$$

Since $m < \infty$ the power series for $R(x)$ converges absolutely and uniformly in $|x| \leqq 1$. We claim that $R(x)$ has no zeros for $|x| \leqq 1$. For $|x| < 1$ this is clear from (2), since $P(x)$ has positive coefficients

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and therefore $|P(x)|<1$, for $|x|<1$. Then any zeros of $R(x)$ must occur on the circle $|x|=1$. From (1) it follows that $R(1)=m\neq 0$. Hence any zero must be of the form $x_0=e^{i\theta_0}$, $0<\theta_0<2\pi$. If $R(x_0)=0$, then (2) implies $P(e^{i\theta_0})=1$. Since $p_k\geqq 0$ this can happen only if $\cos\theta_0 k=1$ for all k for which $p_k\neq 0$. But this is impossible because $P(x)$ would be a power series in x^t, for some integer $t>1$.

Then the function

$$\frac{1-x}{1-P(x)}=\frac{1}{R(x)}$$

has no singularities in $|x|<1$ and we can expand it in a power series

$$\frac{1}{R(x)}=\sum_{n=0}^{\infty}a_nx^n, \tag{3}$$

where

$$a_n=\frac{1}{2\pi i}\int_{|x|=r}\frac{x^{-n-1}}{R(x)}\,dx, \qquad r<1.$$

Now $[R(x)]^{-1}$ is bounded in $|x|<1$; therefore we can apply Lebesgue's convergence theorem to let $r\to 1$; we obtain

$$a_n=\frac{1}{2\pi i}\int_{|x|=1}\frac{x^{-n-1}}{R(x)}\,dx=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{-in\theta}}{R(e^{i\theta})}\,d\theta, \tag{4}$$

for $n=0, 1, \cdots$.

But we have already seen that

$$R(e^{i\theta})=\sum_{n=0}^{\infty}r_ne^{i\theta n}$$

converges absolutely and has no zeros. It follows by a theorem of Wiener[2] that $[R(e^{i\theta})]^{-1}$ has an absolutely convergent expansion

$$\frac{1}{R(e^{i\theta})}=\sum_{n=-\infty}^{\infty}b_ne^{in\theta},$$

where $\sum|b_n|<\infty$ and

$$b_n=\int_{-\pi}^{\pi}\frac{e^{-in\theta}}{R(e^{i\theta})}\,d\theta, \qquad -\infty<n<\infty.$$

A comparison of this and (4) shows that $\sum|a_n|<\infty$. From (3) we

[2] A. Zygmund, *Trigonometrical series*, Warsaw, 1935, p. 140.

can conclude that $\sum a_n = m^{-1}$. But

$$\sum_{n=0}^{\infty} a_n x^n = (1 - x) \sum_{k=0}^{\infty} u_k x^k.$$

It follows from Abel's theorem that $u_k \to 1/m$.

This argument does not work in the case $m = \infty$ since then $R(x)$ is not bounded. The following proof is quite elementary and does not distinguish between the two cases.

Clearly

$$u_n = p_0 u_n + p_1 u_{n-1} + \cdots + p_{n-1} u_1 + p_n u_0, \qquad n \geqq 1.$$

Moreover, since $R(x)U(x) = (1-x)^{-1}$, we have

$$r_0 u_n + r_1 u_{n-1} + \cdots + r_{n-1} u_1 + r_n u_0 = 1.$$

Let

$$\lambda = \limsup u_n,$$

and let $\{n_\nu\}$ be a sequence such that $u_{n_\nu} \to \lambda$. We claim that for any fixed $j > 0$ for which $p_j > 0$ we have $u_{n_\nu - j} \to \lambda$. In fact, assume that $u_{n_\nu - j} \to \lambda' < \lambda$. Then for sufficiently large ν

$$\begin{aligned} \lambda - \epsilon < u_{n_\nu} & \\ &< (p_0 + \cdots + p_{j-1} + p_{j+1} + \cdots + p_N)(\lambda + \epsilon) + p_j \lambda' + \epsilon \\ &\leqq (1 - p_j)(\lambda + \epsilon) + p_j \lambda' + \epsilon < \lambda - p_j(\lambda - \lambda') + 2\epsilon \end{aligned}$$

whence $\lambda' = \lambda$.

Repeating the same argument, we see that

$$\lim_{\nu \to \infty} u_{n_\nu - sj} = \lambda \tag{5}$$

for every fixed integer $j \geqq 0$, provided only that $u_{n_\nu} \to \lambda$ and $p_j > 0$. Now consider the set of all integers j for which $p_j > 0$. By hypothesis, their greatest common divisor is 1. We can, therefore, find a *finite* collection $a_1, a_2, \cdots, a_t$ of subscripts such that $p_{a_1} > 0, \cdots, p_{a_t} > 0$ and that their greatest common divisor is 1. Then by (5)

$$\lim_{\nu \to \infty} u_{n_\nu - k} = \lambda \tag{6}$$

for every fixed integer k of the form

$$k = x_1 a_1 + x_2 a_2 + \cdots + x_t a_t. \tag{7}$$

However, *every* integer $k > a_1 a_2 \cdots a_t$ can be put into the form (7) and hence (6) holds for every sufficiently large fixed k. Now put in (5) $n=N_\nu=n_\nu-a_1a_2\cdots a_t$. Then for every fixed M

$$1 \geqq r_0 u_{N_\nu} + r_1 u_{N_\nu-1} + \cdots + r_M u_{N_\nu-M}.$$

As $\nu\to\infty$ all terms $u_{N_\nu-k}\to\lambda$ and hence

$$1 \geqq \lambda(r_0 + r_1 + \cdots + r_M)$$

or $\lambda \leqq 1/m$ (with $\lambda=0$ if $\sum r_n=\infty$).

If $m<\infty$ we can use a similar argument for $\mu=\lim\inf u_n$ to show that $\mu\geqq 1/m$. This proves the theorem.

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A CONSISTENCY THEOREM

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1. Introduction. Of primary importance in a theory of representation of functions by series which do not necessarily converge is its consistency theorem, which states that if a series which represents a function F converges to a function Φ, then $F\equiv\Phi$. Such a theorem for asymptotic representation in a strip region of a function by Dirichlet series with a certain logarithmic precision, an idea introduced by Mandelbrojt [1],[2] is the subject matter of this note. From it follow similar theorems for less general extensions of the idea of asymptotic series. The method consists in using the proof of the fundamental theorem in [1] to set up a homogeneous linear differential equation of infinite order with constant coefficients, which must be satisfied by the difference $F-\Phi$; then applying a method of Ritt to show that the only solution is identically zero.

The notation used by Mandelbrojt in [1] will be used here also. Let $\{\lambda_n\}$ be an increasing sequence of positive numbers $(0<\lambda_n\uparrow)$. Denote by $N(\lambda)$, defined for $\lambda>0$, the *distribution function* of $\{\lambda_n\}$; that is, the number of terms in the sequence $\{\lambda_n\}$ less than λ; and by $D(\lambda)$ the *density function* of $\{\lambda_n\}$: $D(\lambda)=N(\lambda)/\lambda$. Let $D^{\cdot}$ represent the *upper density*: $D^{\cdot}=\lim\sup_{\lambda\to\infty}D(\lambda)$; and $D^{\cdot}(\lambda)$ the *upper density function* of $\{\lambda_n\}$: $D^{\cdot}(\lambda)=\text{l.u.b.}_{x\geqq\lambda}D(x)$; clearly $D^{\cdot}(\lambda)$ is continuous and decreases to $D^{\cdot}$ (unless $D^{\cdot}(\lambda)\equiv D^{\cdot}=\infty$).

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[1] The author is indebted to Professor Mandelbrojt for suggesting the problem considered in this note.

[2] Numbers in brackets refer to the bibliography at the end of the paper.