# ON THE EXTENSION OF A TRANSFORMATION

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**0. Introduction.** In a problem on surface area the writer and Helsel[1] were confronted with the following question. Can a Lipschitzian transformation from a set in a Euclidean three-space into a Euclidean three-space be extended to a Lipschitzian transformation defined on the whole space? The affirmative answer to this question has been given by Kirszbraun.[2] In fact, Kirszbraun shows this result for any Euclidean spaces (see also Valentine).[3] In studying these papers the writer noted that a more general extension problem could be formulated and a different method of proof to the problem could be obtained. To formulate the more general problem we first give some definitions.

Let $M$ be a metric space, the distance between two points $p_1, p_2 \in M$ being denoted by $p_1p_2$. Let $\mathcal{P}(M)$ be the class of real-valued continuous functions $g(t)$, $0 \leqq t < \infty$, which satisfy the conditions: (a) $g(0)=0$, (b) $g(t)>0$ for $t>0$, (c) for any finite number of points $p_0, p_1, \cdots, p_m$ in $M$ the real quadratic form $\sum_{i,j=1}^{m} [g(p_0p_i)^2+g(p_0p_j)^2-g(p_ip_j)^2]\xi_i\xi_j$ is positive. Let $g(t) \in \mathcal{P}(M)$. A transformation $p^*=\phi(p)$ from a set $E$ in $M$ into a metric space $M^*$ will be said to satisfy the condition $C(g)$ on $E$ if, for every pair of points $p_1, p_2 \in E$, $p_1^*p_2^* \leqq g(p_1p_2)$, where $p_i^*=\phi(p_i)$, $i=1, 2$. We shall say that $\phi(p)$ can be extended to a set $E'$, $E \subset E' \subset M$, preserving the condition $C(g)$ if there exists a transformation $p^*=\Phi(p)$ from $E'$ into $M^*$ which satisfies the condition $C(g)$ on $E'$ and is equal to $\phi(p)$ on $E$.

In this paper we prove the following result. Let $M$ be a separable metric space and let $g(t) \in \mathcal{P}(M)$. Then any transformation from a set $E$ in $M$ into a Euclidean space which satisfies the condition $C(g)$ on $E$ can be extended to $M$ preserving the condition $C(g)$.

We give two examples to illustrate this result. We shall use the vector notation $x$ to represent a point in a Euclidean $n$-space $E_n$, and we shall denote by $|x_1-x_2|$ the distance between two points $x_1, x_2$. Let $x_0, x_1, \cdots, x_m$ be $m+1$ points in $E_n$ and let $\xi_1, \cdots, \xi_m$ be $m$ real numbers. From the relation $(x_i-x_j)^2=(x_0-x_i)^2+(x_0-x_j)^2-2(x_0-x_i)(x_0-x_j)$, the square of the vector $x=L\xi_1(x_0-x_1)+ \cdots$

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[1] Helsel and Mickle, Bull. Amer. Math. Soc. vol. 54 (1948) pp. 235–238.

[2] Kirszbraun, Fund. Math. vol. 22 (1934) pp. 77–108.

[3] Valentine, Bull Amer. Math. Soc. vol. 49 (1943) pp. 100–108.

$+L\xi_m(x_0-x_m)$, $L>0$, is given by

$$x^2 = L^2 \sum_{i,j=1}^{m} (x_0 - x_i)(x_0 - x_j)\xi_i\xi_j$$

$$= \frac{L^2}{2} \sum_{i,j=1}^{m} [\,|x_0 - x_i|^2 + |x_0 - x_j|^2 - |x_i - x_j|^2]\xi_i\xi_j \geqq 0.$$

Thus the function $g(t)=Lt$, $L>0$, is in $\mathcal{P}(E_n)$ and the results of Kirszbraun follow as a special case of the results of this paper. Schoenberg[4] has shown that the function $g(t)=Lt^\alpha$, $L>0$, $0<\alpha\leqq 1$, is in $\mathcal{P}(E_n)$. Thus a transformation from a set in a Euclidean space into a Euclidean space which satisfies a Lipschitz-Hölder condition ($L>0$, $0<\alpha\leqq 1$) can be extended to the whole space preserving this same Lipschitz-Hölder condition.

**1. Preliminary remarks.** In this section we give some well known concepts and lemmas for a Euclidean $n$-space $E_n$. A set is called convex if the line segment joining any two points of the set is in the set. If $E$ is a closed convex set and $x\notin E$, then there is a unique point $x^*\in E$ which is closest to $x$. (Since $E$ is closed there is one such point. If there were two, the midpoint of the line segment joining them would be in $E$ and closer to $x$ than either of them.) For a finite set of points $x_1, \cdots, x_m$, we denote by $V(x_1, \cdots, x_m)$ the smallest convex set containing them. $V(x_1, \cdots, x_m)$ is a closed set consisting of those points given by the relation $x=c_1x_1+\cdots+c_mx_m$, where the $c_i$'s are non-negative and $c_1+\cdots+c_m=1$.

LEMMA 1.1. *Let $E$ be a closed convex set, $x_0$ a point not in $E$, $x_0^*$ the unique point of $E$ closest to $x_0$, and $y=tx_0+(1-t)x_0^*$, $0\leqq t<1$. Then $|y-x|<|x_0-x|$ for every point $x\in E$.*

PROOF. Since $|y-x|\leqq t|x_0-x|+(1-t)|x_0^*-x|$, $t\neq 1$, it is sufficient to prove that $|x_0^*-x|<|x_0-x|$ for all $x\in E$. Assume there is a point $x_1\in E$ for which $|x_0^*-x_1|\geqq|x_0-x_1|$. Then the numbers $a=|x_0-x_0^*|$, $b=|x_0-x_1|$, $c=|x_0^*-x_1|$ satisfy the inequalities $a<b\leqq c$ and the number $t^*=(a^2+c^2-b^2)/2c^2$ satisfies the inequalities $0<t^*<1/2$. Thus the point $x=t^*x_1+(1-t^*)x_0^*$ is in $E$ and $|x_0-x|^2=|x_0-x_0^*|^2-t^{*2}c^2<a^2$, contradicting the assumption that $x_0^*$ is the point of $E$ closest to $x_0$.

LEMMA 1.2. *Let $\Sigma$ be a set of closed spheres in $E_n$ such that there is no point in common to all of them. Then there is a finite set of spheres in $\Sigma$*

[4] Schoenberg, Ann. of Math. vol. 38 (1937) pp. 787–793; Amer. J. Math. vol. 67 (1945) pp. 83–93.

*which have no point in common.*[5]

PROOF. Let $S_0$ be one of the spheres in $\Sigma$. Then the sum of the complements of the remaining spheres cover $S_0$. Since this is an open covering of $S_0$, there is a finite number of spheres $S_1, \cdots, S_m$ in $\Sigma$ the sum of whose complements covers $S_0$. Hence, $S_0, S_1, \cdots, S_m$ have no point in common.

**2. Lemmas.** For a given integer $m$, let $a_1, \cdots . a_m$ be a given set of $m$ positive numbers and let $x_1, \cdots, x_m$ be a given set of (not necessarily distinct) $m$ points in $E_n$. For each point $x$ let

$$f(x) = \max (| x - x_i | /a_i), \qquad i = 1, \cdots, m. \tag{2.1}$$

$f(x)$ is obviously a continuous function of $x$ and there exists a point $x_0$ for which $f(x_0) = \min f(x)$. We have the following result concerning the location of a point $x_0$ at which $f(x)$ assumes a minimum value.

LEMMA 2.1. *Let $x_0$ be a point such that $f(x_0) = \min f(x)$ and let $x_{m_1}, \cdots x_{m_k}$ be all the points in the set of points $x_1, \cdots, x_m$ for which the equality*

$$f(x_0) = | x_0 - x_i | /a_i \tag{2.2}$$

*holds. Then $x_0 \in V(x_{m_1}, \cdots, x_{m_k})$.*

PROOF. Assume $x_0 \notin V(x_{m_1}, \cdots, x_{m_k})$. Let $x_0^*$ be the unique point of $V(x_{m_1}, \cdots, x_{m_k})$ closest to $x_0$. For each integer $j$ let $y_j = t_j x_0 + (1-t_j)x_0^*$, $0 \leqq t_j < 1$, $t_j \to 1$, for $j \to \infty$. By Lemma 1.1, $|y_j - x| < |x_0 - x|$ for every point $x \in V(x_{m_1}, \cdots, x_{m_k})$. Thus, since $f(x_0) \leqq f(y_j)$, $f(y_j) = |y_j - x_i| /a_i$, $1 \leqq i \leqq m$, for some $x_i \notin V(x_{m_1}, \cdots, x_{m_k})$. There are an infinite number of the $x_i$'s corresponding to the points $y_j$ which are the same and we can assume without loss of generality that all of them are the same point $x_{m_{k+1}}$, $1 \leqq m_{k+1} \leqq m$. Since $f(y_j) \to f(x_0)$ for $j \to \infty$, $f(x_0) = \lim |y_j - x_{m_{k+1}}| /a_{m_{k+1}} = |x_0 - x_{m_{k+1}}| /a_{m_{k+1}}$, $x_{m_{k+1}} \notin V(x_{m_1}, \cdots, x_{m_k})$. Thus (2.2) holds for $i = m_{k+1}$. This contradicts the fact that (2.2) holds only for the points $x_{m_1}, \cdots, x_{m_k}$. Hence $x_0 \in V(x_{m_1}, \cdots, x_{m_k})$.

We now prove the fundamental lemma of the paper.

LEMMA 2.2. *Let $M$ be a metric space, let $g(t) \in \mathcal{P}(M)$ and let $p_1, \cdots, p_m$ be $m$ distinct points and $x_1, \cdots, x_m$ be $m$ points in $M$ and $E_n$ respectively for which the inequalities $|x_i - x_j| \leqq g(p_i p_j)$, $i, j = 1, \cdots, m$, hold. Then, for any point $p_0 \in M$, there exists a point*

[5] We use the term closed sphere to mean the set of points $x$ which satisfy the inequality $|x - x_0| \leqq r$ for fixed $x_0$ and $r > 0$.

*$x_0 \in E_n$ for which the inequalities $|x_0 - x_i| \leqq g(p_0 p_i)$, $i = 1, \cdots, m$, hold.*

PROOF. If $p_0 = p_i$, $1 \leqq i \leqq m$, let $x_0 = x_i$. Assume $p_0 \neq p_1, \cdots, p_m$. Set $a_i = g(p_0 p_i)$, $i = 1, \cdots, m$. Since $g(t) \in \mathcal{P}(M)$, each $a_i > 0$. If $x_0$ is a point for which $f(x_0) = \min f(x)$ (see (2.1)), we assert that $\lambda = f(x_0) \leqq 1$. That is to say, $|x_0 - x_i| \leqq g(p_0 p_i)$, $i = 1, \cdots, m$. If $\lambda = 0$, then $\lambda \leqq 1$. Assume $\lambda > 0$. Set $a_{ij} = g(p_i p_j)$, $b_{ij} = |x_i - x_j|$ and $b_i = |x_0 - x_i|$, $i, j = 1, \cdots, m$. By renumbering if necessary, let $x_1, \cdots, x_k$ be the points for which the equality $f(x_0) = b_i / a_i = \lambda$ holds. By Lemma 2.1, $x_0 \in V(x_1, \cdots, x_k)$. Thus we have non-negative numbers $c_1, \cdots, c_k$, $c_1 + \cdots + c_k = 1$ such that $x_0 = c_1 x_1 + \cdots + c_k x_k$ or $c_1(x_0 - x_1) + \cdots + c_k(x_0 - x_k) = 0$. By squaring this expression and using the relation $(x_i - x_j)^2 = (x_0 - x_i)^2 + (x_0 - x_j)^2 - 2(x_0 - x_i)(x_0 - x_j)$ we obtain

$$(2.3)\ \sum_{i,j=1}^{k} (x_0 - x_i)(x_0 - x_j) c_i c_j = \frac{1}{2} \sum_{i,j=1}^{k} (b_i^2 + b_j^2 - b_{ij}^2) c_i c_j = 0.$$

Since $\lambda > 0$, $x_0 \neq x_1, \cdots, x_k$, and hence at least two of the $c_i$'s are different from zero. Since $g(t) \in \mathcal{P}(M)$, the quadratic form $\sum_{i,j=1}^{k} (a_i^2 + a_j^2 - a_{ij}^2) \xi_i \xi_j$ is positive. Setting $\xi_i = \lambda c_i$, $i = 1, \cdots, k$, and using the fact that $\lambda a_i = b_i$, $i = 1, \cdots, k$, we obtain

$$(2.4)\ \frac{1}{2} \sum_{i,j=1}^{k} (a_i^2 + a_j^2 - a_{ij}^2) \lambda^2 c_i c_j = \frac{1}{2} \sum_{i,j=1}^{k} (b_i^2 + b_j^2 - \lambda^2 a_{ij}^2) c_i c_j \geqq 0.$$

Subtracting (2.3) from (2.4) gives

$$(2.5)\qquad \frac{1}{2} \sum_{i,j=1}^{k} (b_{ij}^2 - \lambda^2 a_{ij}^2) c_i c_j \geqq 0.$$

Since $a_{ij} = b_{ij} = 0$ for $i = j$, the $c_i$'s are non-negative and at least two of the $c_i$'s are different from zero, it follows from (2.5) that $b_{ij}^2 - \lambda^2 a_{ij}^2 \geqq 0$ for some pair of integers $i$, $j$ with $i \neq j$. For this pair of integers $1 \leqq i, j \leqq k$, $i \neq j$, we have $g(p_i p_j)^2 \geqq |x_i - x_j|^2 = b_{ij}^2 \geqq \lambda^2 a_{ij}^2 = \lambda^2 g(p_i p_j)^2$. Hence $\lambda \leqq 1$. Thus a point $x_0$ at which $f(x)$ assumes a minimum satisfies the conditions of the lemma.

LEMMA 2.3. *Let $M$ be a metric space, let $g(t) \in \mathcal{P}(M)$, let $x = \phi(p)$ be a transformation from a set $E$ in $M$ into $E_n$ and let $p_0$ be any point in $M$. Then, if $\phi(p)$ satisfies the condition $C(g)$ on $E$, $\phi(p)$ can be extended to $E + p_0$ preserving the condition $C(g)$.*

PROOF. If $p_0 \in E$ the extension is immediate. Assume $p_0 \notin E$. For each $p \in E$, let $S_p$ be the set of points $x \in E_n$ which satisfy the inequality $|x-\phi(p)| \leqq g(p_0p)$. Since $g(t) \in \mathfrak{P}(M)$, each $S_p$ is a closed sphere in $E_n$. Assume that there is no point in common to all the spheres. By Lemma 1.2 there is a finite number of these spheres which have no point in common. This contradicts Lemma 2.2. Hence, there is at least one point $x_0$ in all the spheres $S_p$, $p \in E$. Then $\Phi(p_0) = x_0$, $\Phi(p) = \phi(p)$, $p \in E$, is an extension of $\phi(p)$ to $E+p_0$ preserving the condition $C(g)$.

## 3. The main result.

We now state and prove the main result of this paper.

THEOREM. *Let $M$ be a separable metric space, let $g(t) \in \mathfrak{P}(M)$ and let $x = \phi(p)$ be a transformation from a set $E$ in $M$ into a Euclidean space $E_n$. Then if $\phi$ satisfies the condition $C(g)$ on $E$, $\phi$ can be extended to $M$ preserving the condition $C(g)$.*

PROOF. Let $D$ be a finite or denumerable set which is dense in $M$. By Lemma 2.3, $\phi(p)$ can be extended to $E$ plus any point of $D$ and by induction to $E+D$ preserving the condition $C(g)$. Let $x = \Phi(p)$, $p \in E+D$ be the extended transformation. Since $E_n$ is complete and $g(t) \in \mathfrak{P}(M)$, a convergent sequence of points $p_m \in E+D$, $m = 1, 2, \cdots$, determines a convergent sequence of points $\Phi(p_m)$ in $E_n$. Since $E+D$ is dense in $M$, $\Phi(p)$ can be extended to $M$ preserving the condition $C(g)$ in one and only one way.

## 4. Additional remarks.

The writer is indebted to the referee for pointing out the following facts. Any finite set of points in a unitary space is isometrically equivalent to a set of points in some Euclidean space. Hence Lemma 2.2 is valid in any unitary space. Lemma 1.2 is valid in any complete unitary space (see Murray[6] for the case where the space is separable and Alaoglu[7] for the general case). Hence Lemma 2.3 is valid if $E_n$ is replaced by a complete unitary space. Then the theorem in §3 with $E_n$ replaced by a complete unitary space $U$ and with $M$ not assumed to be separable follows from Lemma 2.3 (with $E_n$ replaced by $U$) by applying Zorn's lemma or transfinite induction.

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[6] F. J. Murray, *Linear transformations in Hilbert space*, Princeton University Press, 1941.

[7] Alaoglu, Ann. of Math. vol. 41 (1940) pp. 252–267.