SEMILATTICES AND A TERNARY OERATION IN MODULAR LATTICES

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Before discussing the subject matter proper it is necessary to introduce the following:[1]

LEMMA 1. *The inequality*

$$(1)\quad \{(x\cap(y\cap(v\cup z))\}\cup(v\cap z)\subseteq v\cup(y\cap(x\cup z))\cup(x\cap z)$$

is identically satisfied in any lattice.

PROOF. $x\cap y\cap(v\cup z)\subseteq x\cap y\subseteq(x\cup z)\cap y\subseteq v\cup(y\cap(x\cup z))$,

$$v\cap z\subseteq v\subseteq v\cup(y\cap(x\cup z))$$

and from these two inequalities follows

$$(x\cap y\cap(v\cup z))\cup(v\cap z)\subseteq v\cup(y\cap(x\cup z))$$
$$\subseteq v\cup(y\cap(x\cup z))\cup(x\cap z).$$

For purposes of facility of expression the concept of *semilattice* is here introduced following Klein-Barmen [1]:[2]

DEFINITION 1. A semilattice L_s is a partially ordered system in which a relation $x\sigma y$ is defined which satisfies

S1: For all x, $x\sigma x$,

S2: If $x\sigma y$ and $y\sigma x$, then $x=y$,

S3: If $x\sigma y$ and $y\sigma z$, then $x\sigma z$,

and in which any two elements x and y have a greatest lower bound or meet xmy.

It then follows that xmy or any binary operation xoy which is closed, idempotent, commutative and associative defines, by means of the convention that $x\sigma y$ means $xmy=x$ or $xoy=x$, a semilattice L_s in which xmy or xoy is the greatest lower bound of x and y.

LEMMA 2. *The ternary operation*

$$(2)\quad [x,t,y]=(x\cap(t\cup y))\cup(t\cap y)=(x\cup(t\cap y))\cap(t\cup y)$$

on the elements of a modular lattice L is closed and is an idempotent and

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[1] The author is indebted to Garrett Birkhoff for the proof of Lemma 1, and for helpful criticism.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

associative operation for a constant t. The expression $[x, t, y]$, *for* $t=\text{const.}$, *is said to determine the "operational plane" t.*

(3) The *idempotent law*

$$[x, t, x] = x \text{ for all } x \text{ and } t$$

holds because of the absorption law in L:

$$[x, t, x] = (x \cap (t \cup x)) \cup (t \cap x) = x \cup (t \cap x) = x.$$

The proof of the *associative law* is somewhat longer, proceeding as follows:

Expanding the expression of the associative law

$$[[x, t, y], t, z] = [x, t, [y, t, z]] \tag{4}$$

one obtains

$$\begin{aligned} \{\{x \cap (t \cup y)) \cup (t \cap y)\} \cap (t \cup z)\} \cup (t \cap z) \\ = x \cap \{t \cup \{(y \cap (t \cup z)) \cup (t \cap z)\}\} \\ \cup \{t \cap \{(y \cap (t \cup z)) \cup (t \cap z)\}\}. \end{aligned}$$

Some of the expressions on the right-hand side may be simplified by employing the absorption law and Dedekind's modular identity.

$$\begin{aligned} t \cup \{(y \cap (t \cup z)) \cup (t \cap z)\} &= t \cup (t \cap z) \cup (y \cap (t \cup z)) \\ &= t \cup (y \cap (t \cup z)) = (t \cup y) \cap (t \cup z), \\ t \cap \{(y \cap (t \cup z)) \cup (t \cap z)\} &= t \cap (t \cup z) \cap (y \cup (t \cap z)) \\ &= t \cap (y \cup (t \cap z)) = (t \cap y) \cup (t \cap z). \end{aligned}$$

Therefore,

$$\begin{aligned} \{\{(x \cap (t \cup y)) \cup (t \cap y)\} \cap (t \cup z)\} \cup (t \cap z) \\ = \{x \cap (t \cup y) \cap (t \cup z)\} \cup \{(t \cap y) \cup (t \cap z)\}. \end{aligned}$$

Putting $X = x \cap (t \cup y)$, $Y = t \cup z$, $U = t \cap y$, $V = t \cap z$ where $V \subseteq Y$ and $U \subseteq t \subseteq Y$, the above formula becomes

$$\{(X \cup Y) \cap Y\} \cup V = (X \cap Y) \cup (U \cap V)$$

and, in view of $(X \cup U) \cap Y = (U \cup X) \cap Y = U \cup (X \cap Y)$,

$$\{U \cup (X \cap Y)\} \cup V = (X \cap Y) \cup (U \cup V)$$

which is an identity, thus concluding the proof.

An immediate consequence, then, is:

THEOREM 1. *The commutative "products"* $[x, t, y]$ *for a constant t,* *that is, those which satisfy* $[x, t, y]=[y, t, x]$, *form a semilattice in which the element* $[x, t, y]$ *is the greatest lower bound of* x *and* y.

THEOREM 2. *Whether commutative or not, the "product"* $[x, t, y]$ *is always determined for a modular lattice L and in the ternary operational system thus obtained any two operational planes* u *and* v *satisfy the following identical equation, namely,*

$$[[x, u, [y, v, z]], v, [y, u, z]] = [[x, v, [y, u, z]], u, [y, v, z]]. \tag{5}$$

To prove this formula, it is expanded by means of (2) above and is subsequently shown to be an identity. Putting

$$\begin{aligned} Y &= u \cup (y \cap (v \cup z)) \cup (v \cap z), \\ Z &= u \cap ((y \cap (v \cup z)) \cup (v \cap z)), \\ U &= v \cup (y \cap (u \cup z)) \cup (u \cap z), \\ V &= v \cap ((y \cap (u \cup z)) \cup (u \cap z)) \end{aligned}$$

where $Z \subseteq Y$, $V \subseteq U$ and, in view of Lemma 1, $Z \subseteq U$ and $V \subseteq Y$, the equation (5) becomes

$$(((x \cap Y) \cup Z) \cap U) \cup V = (((x \cap U) \cup V) \cap Y) \cup Z$$

or, in view of Dedekind's modular identity,

$$(V \cup (x \cap Y) \cup Z) \cap U = (Z \cup (x \cap U) \cup V) \cap Y.$$

Since $Z \subseteq U$ and $V \subseteq U$ give $Z \cup V \subseteq U$ and, similarly, $Z \subseteq Y$ and $V \subseteq Y$ give $Z \cup V \subseteq Y$, the last equation becomes, in view of Dedekind's modular identity,

$$Z \cup V \cup (x \cap Y \cap U) = Z \cup V \cup (x \cap U \cap Y).$$

(5) is thus proven to be an identity.

EXAMPLE. Designating the elements of the nondistributive modular lattice L_5 by 0 (least), a, b, c, e (greatest), the commutative "products" of the operational plane $[x, a, y]$ define a semilattice which is not a lattice, similar remarks applying to $[x, b, y]$ and $[x, c, y]$. The noncommutative products, namely, $[b, a, c]=b$, $[c, a, b]=c$, $[a, b, c]=a$, $[c, a, b]=c$, $[a, c, b]=a$, $[b, c, a]=b$ do not belong to the semilattices.

Being partially ordered systems, semilattices may be represented by diagrams. In L_5 links are preserved in all semilattices defined by (2) with constant t; it is the author's conjecture that this rule holds for the semilattices defined in any modular lattice.

When a lattice is distributive in addition to being modular, the expression (2) becomes

$$[x, t, y] = (x \cap t) \cup (t \cap y) \cup (y \cap x). \tag{6}$$

This is the ternary operation (x, t, y) which was independently introduced by Grau[3] [2] for Boolean algebras and by Birkhoff and Kiss [3] for distributive lattices in general.

It is obvious from the expression (2) of $[x, t, y]$ that $[x, t, y] = [x, y, t]$; on the other hand the above example shows that, in some cases at least, $[x, t, y] \neq [y, t, x]$ and also $[x, t, y] \neq [t, x, y]$.

Complementation in distributive lattices has been defined by Birkhoff and Kiss [3] and can now be extended to modular lattices by means of the following:

DEFINITION 2. The elements x, x' of a modular lattice L are called "strictly complementary" if and only if

$$[x, t, x'] = t \text{ for all } t. \tag{7}$$

THEOREM 3. *Strict complementation in a modular lattice is unique.*

PROOF. If x has two complements, x' and x'', then $x'' = [x, x'', x'] = [x, x', x''] = x'$.

THEOREM 4. *The* 0 (*least*) *and* e (*greatest*) *elements of a modular lattice are always strictly complementary*; *furthermore, the* $[x, 0, y]$ *and* $[x, e, y]$ *planes of the ternary lattice give the* $x \cap y$ *and* $x \cup y$ *operations, respectively.*

PROOF. $[0, t, e] = (0 \cap (t \cup e)) \cup (t \cap e) = 0 \cup t = t$,

$$[x, 0, y] = (x \cap (0 \cup y)) \cup (0 \cap y) = (x \cap y) \cup 0 = x \cap y,$$

$$[x, e, y] = (x \cap (e \cup y)) \cup (e \cap y) = (x \cap e) \cup y = x \cup y.$$

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3. Garrett Birkhoff and S. A. Kiss, *A ternary operation in distributive lattices*, Bull. Amer. Math. Soc. vol. 58 (1947) pp. 749–752.

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[3] Grau uses the notation $x^t y$ for (x, t, y).