$$\iint_{\mathfrak{D}} H(t(w)) \,|\, \mathfrak{J}_e(w) \,| = \iint H(z)\kappa(z, \mathfrak{D})$$

holds whenever either integral exists. Also, if $\nu(z)$ is the sum of $i_e(w)$ for all inverse images w of z, then

$$\iint_{\mathfrak{D}} H(t(w))\mathfrak{J}_e(w) = \iint H(z)\nu(z)$$

if the left member exists.

The family of eAC transformations is not merely larger than the corresponding families in earlier theories of plane transformations; it also has closure properties which simplify the task of establishing that various special types of transformations are actually eAC.

The theory culminates in Part V, devoted to areas of surfaces. The strength of the results attained can be shown by quoting two theorems. The functions $x(u, v)$, $y(u, v)$, $z(u, v)$ $(0 \leqq u \leqq 1,\ 0 \leqq v \leqq 1)$ defining S furnish three plane transformations, by projection. The square root of the sum of the squares of the three essential generalized Jacobians will be denoted by $W_e(u, v)$; if the ordinary Jacobians exist, the square root of the sum of their squares is $W(u, v)$. Then:

(A) If $A(S) < \infty$, the three plane transformations are eBV, and W_e is defined almost everywhere in the unit square and is summable; and its integral is at most $A(S)$, being equal to $A(S)$ if and only if the three plane transformations are eAC.

(B) If $A(S) < \infty$ and $W(u, v)$ is defined almost everywhere in the unit square, it is summable, and its integral is at most $A(S)$, being equal to $A(S)$ if and only if the three plane transformations are eAC.

By his choice of methods of proof and by clarity of exposition, the author has provided a well-engineered road into a difficult territory. However, in the multitude of theorems a reader might well be puzzled about the interrelations, motivations and origins of the mathematical objects he encounters. The author has therefore provided a final chapter to each part except the first, in which he casts a backward look over the preceding chapters, coordinates their contents, and indicates directions in which further research is needed.

E. J. McShane

The advanced theory of statistics. Vol. 2. By M. G. Kendall. London, Griffin, and Philadelphia, Lippincott. 1st ed., 1946, 2d ed., 1948. 8+521 pp. 50s.

The great treatise, of which the first volume was reviewed in Bull. Amer. Math. Soc. vol. 51 (1945) p. 214, is now complete. For the first time a connected account is available of the main outlines of the whole theory of statistics, from elementary probability and the fundamentals of estimation and testing of hypotheses to recent advanced researches and techniques, with attention to many unsolved problems. A debt of gratitude is due the author for the prodigious task of working through thousands of papers scattered through a multitudinous variety of journals devoted to statistics, mathematics, economics, psychology, actuarial work, physics, biology of many kinds, and other fields, and assembling, arranging and rewriting their contents with original additions and comments.

The second volume covers many of the more difficult and controversial subjects concerning which knowledge is still incomplete. This was apparently one object in the ordering of the chapters of the complete treatise, though both historical and logical bases for the arrangement are also discernible, and the actual arrangement seems to be a compromise. Volume 2 begins with Chapters 17 on the method of maximum likelihood, 18 on estimation in general, 19 on confidence intervals, 20 on fiducial inference, and 21 on some common tests of significance. All these constitute chiefly an expansion of R. A. Fisher's work, with more mathematical details than Fisher himself gives, and with accounts of the extensions of Fisher's ideas by Doob, Welch, Pitman, Bartlett, Geary, Koopman, Dugué, Wilks, Wald, Daly, E. S. Pearson, Yates, Sukhatme and others. Chapter 22, on regression, includes an account of the orthogonal polynomials used for fitting with equidistant intervals of the argument. The next two chapters, on the analysis of variance, and Chapter 25 on the design of sampling inquiries, again reflect strongly Fisher's influence, though Mr. Kendall's interest as an economic statistician leads to a greater emphasis on social-economic sampling than on the agricultural and biological problems which originally led to the modern theory of experimental design as formulated by Fisher.

The work of Neyman and Pearson on the fundamentals of testing hypotheses is postponed until after all this, with the result that its assimilation with Fisher's work into a rounded theory is incomplete. However the treatment is excellent, and many a reader will get his first clear idea of the subject from these two chapters. The difficult problems of testing hypotheses involving two or more parameters are expounded as adequately as the present stage of knowledge permits.

The chapter on multivariate analysis covers most of the fundamental work in this field, including Wishart's distribution, the T distribution, discriminatory analysis, and canonical correlations. It is not, however, made clear that the distribution found for the roots of a determinantal equation is applicable to the comparison of two samples as well as to canonical correlations; and a fuller account would be desirable of the contributions of Wilks and Hsu to the generalized analysis of variance.

The two final chapters deal with time series. This vexing but urgent subject is still not in a satisfactorily complete state. Mr. Kendall has given the principal available material a thorough working over, using a variety of series of actual observations and a series of random numbers as examples for the application of weighted and unweighted moving averages, correlograms, and other devices. Interlaced parabolas, the variate difference method, autoregression equations (that is, stochastic difference equations), serial correlation, lag correlation and periodogram analysis are all treated, and outstanding unsolved problems are indicated.

Mr. Kendall amplifies R. A. Fisher's discussion of fiducial probability, including the Behrens test for the difference between two means. In attempting to understand these ideas and connect them with other statistical tests one encounters some fundamental difficulties. Probability is in the conception of Fisher and nearly all other present-day statisticians an idealization of frequency-ratios in large samples, and is thus subject to operational measure and verification. This concept should be distinguished sharply from the "degrees of rational belief" of writers such as Keynes who have used the word "probability" in this sense, a sense to which the reviewer would prefer, in the interest of clarity, to attach the word "credibility." In this review "probability" will be used with the former "objective" or statisticians' meaning. When we speak of probability distributions, and of probability levels for tests of significance, we ordinarily understand probability in this sense. Thus, when the Student distribution is used in a way now familiar to test the hypothesis that a sample has been drawn from a population of mean value 37 and the .05 probability level is employed, with the certain knowledge that the sample is a random one from some normal population, we can make the following statement: In the class of cases in which the mean of the population is 37, this hypothesis will be erroneously rejected with probability exactly .05. This statement implies an anticipation that in a long run of such trials the erroneous rejection will take place in almost exactly one-twentieth of the cases. Experiments of just this type have in fact been made, and have yielded close agreement with theory. Further consequences of routine application of the Student test have been elucidated through the power function and certain theorems on optimum properties of tests. The growing popularity of such tests is based on considerations of this kind. "Belief" in the hypothesis that the true mean is 37 may be determined by the outcome of the Student test in a way wholly consistent with the pragmatic philosophy of William James, according to which the truth is that which works, and the goodness of a belief is to be judged by its total consequences, in this case its probable consequences with measurable probabilities.

The "fiducial probability" and "fiducial distributions" introduced by Fisher and now expounded by Kendall do not refer to probability in the sense indicated above. This fact, which has been lost sight of in much discussion, is pointed out by Kendall on pages 90 and 112. Furthermore, if one tries to read into the derivation of the Behrens-Fisher test on pages 91 and 92 the objective or frequency meaning of probability, the result is nonsense. Thus the distribution of

$$\epsilon = t_1 \cos \psi - t_2 \sin \psi$$

is obtained by convolution of two independent Student distributions of t_1 and t_2, with ψ regarded as a constant. But ψ is defined as the arctangent of the ratio of the two sample standard deviations, and if these are held constant t_1 and t_2 do not have the Student distribution but normal distributions, whence the conditional probability distribution of ϵ is normal.

Thus the Behrens distribution does not give probabilities in the ordinary sense, but fiducial probabilities. The application of this test for the difference of two means does not yield definite proportions of fallacious rejections which can be expected to be verified in long series of experiments, but instead a specially defined measure of the degree of belief imputed to the result. What may be the consequences of routine application of the test remain to be elucidated through study of its power function. The underlying principle is in contrast to the pragmatic interpretation of the simpler tests. Thus we have what appears to be a dangerous and objectionable dualism. To escape from such dualism the available alternatives seem to be (a) to modify or abandon the Behrens test, (b) to derive it from some different principle, such for example as the comparison of the sample in hand with a sub-population of possible samples defined by constancy of some function of the sample standard deviations other than their ratio, but using the standard notion of probability, or (c) to insist that the proper justification of all statistical tests is to be obtained through fiducial probability. The last would be a drastic step hard to justify on pragmatic or other established principles, and in any postulational treatment would require an independent axiom not yet precisely stated and whose full consequences have not been explored.

In trying to understand the literature of modern mathematical statistics it is essential to remember that, wherever written, it is fundamentally British, and derives historically from the British tradition of applied mathematics much more than from the meticulous continental tradition of rigorous pure mathematics. Karl Pearson began as a mathematical physicist working in elasticity, R. A. Fisher as an astronomer, "Student" as a chemist, and all became biological statisticians. Like Newton, Stokes, Green and the physicists, English statisticians have been more concerned with getting useful formulae than in delimiting the exact conditions of their validity, and have frequently used loose mathematical arguments. The proofs in the present volume are largely carried over from the original sources, often with arguments in forms subject to criticism. Thus on p. 7, following the formula for a two-dimensional normal distribution regarded as a limiting form, the statement is made and proved that the correlation is unity, meaning that it approaches unity, but this contradicts the postulated two-dimensional density. A more serious difficulty is with Fisher's famous original proof of the efficiency of maximum likelihood estimates, which became the point of departure of much new work and is here reproduced on pages 18 and 19. The free interchange of various limiting processes in the absence of any proof of validity, and the omission from (17.66) of terms involving the derivative of v, are faults in this proof which should be repaired at the first opportunity, even though similar results have recently been obtained by Cramér and Rao using other methods. An error due to the present reviewer, who in 1930 gave a proof of consistency and asymptotic normality of maximum likelihood estimates for discrete distributions but used insufficient care in passing over to continuous distributions, is here reproduced. The essential correctness of the result was established in 1936 by Doob by a different method.

There are some difficulties due in part to oversimplification of notation without sufficient indication as to what arguments enter into particular functions. Thus in various uses of the formula $\phi' = A + B\phi$ in Chap. 27, the reader will do well to remember that A and B may involve the variable parameter. On p. 333 there is a slight error in the statement of the region of variation of the a_{ij}; the matrix must be positive definite. Also, it is still not clear what the statistician is supposed to do with ancillary statistics.

These and similar shortcomings are far more than compensated by the accomplishment of the immense task. The 62-page bibliography of something like 1800 entries (which the author states constitute only half his collection of references) is in itself enough to make this an indespensable work of reference. With the companion Volume 1 (of which a slightly revised third edition has now appeared) it forms a contribution far beyond the range of ordinary textbooks, suitable for reference and for serious students.

HAROLD HOTELLING

The methods of plane projective geometry based on the use of general homogeneous coordinates. By E. A. Maxwell. Cambridge University Press, 1946. 19+230 pages. $2.75.

This is an introductory treatise on algebraic projective geometry. As the author states in the preface, "It is written as a study of *methods* and not as a catalogue of theorems." Synthetic methods are used occasionally when deemed preferable. Although it is stated in the introduction, "No knowledge of geometry is assumed explicitly," it is implied that the student should have had plane analytical geometry, including the properties of conic sections, before beginning the study of this book.

The first ten chapters deal with non-metric concepts. The principal topics and the order in which they occur are: homogeneous coordinates, equations of lines, duality, one-to-one correspondence cross ratio, conics treated parametrically, projective properties of conics, the quadrangle and its dual, the generation of conics by pairs of projective pencils or ranges, pencils of conics, harmonic properties, reciprocation, poristic systems, theorems of Pascal and Brianchon.

In the last two chapters, eleven and twelve, relations between projective and Euclidean geometry are studied both algebraically and synthetically.

The book is well written; the explanations are clear and ample. The printing is excellent and the formulas well displayed. The caption of each section describes pithily the contents. The book has an excellent index and a very detailed table of contents. The total absence of figures, however, is to be deplored; illustrative drawings are especially helpful to a beginner in projective geometry.

The many excellent and carefully chosen problems constitute one of the best features of the book. These problems are chiefly taken from various examinations. At the end of each problem is a symbol referring to the source given in the preface. Answers to the more difficult problems are given in the back of the book.

T. R. HOLLCROFT