ON RINGS OF ANALYTIC FUNCTIONS

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Let D be a domain in the complex plane (Riemann sphere) and $R(D)$ the totality of one-valued regular analytic functions defined in D. With the usual definitions of addition and multiplication $R(D)$ becomes a commutative ring (in fact, a domain of integrity). A one-to-one conformal transformation $\zeta = \phi(z)$ of D onto a domain Δ induces an isomorphism $f \to f^*$ between $R(D)$ and $R(\Delta)$: $f(z) = f^*[\phi(z)]$. An anti-conformal transformation

$$\zeta = \overline{\phi(z)}$$

also induces an isomorphism:

$$\overline{f(\bar{z})} = f^*[\overline{\phi(\bar{z})}].$$

The purpose of this note is to prove the converses of these statements.

THEOREM I. *If $R(D)$ is isomorphic to $R(\Delta)$, then there exists either a conformal or an anti-conformal transformation which maps D onto Δ.*[1]

THEOREM II. *If D and Δ possess boundary points, then every isomorphism between $R(D)$ and $R(\Delta)$ is induced by a conformal or an anti-conformal transformation of D onto Δ.*

Theorem I may be regarded as a complex variable analogue of theorems characterizing a topological space in terms of the family of its continuous functions. If $R(D)$ is made into a topological ring by defining $f_n \to f$ to mean that $f_n(z) \to f(z)$ uniformly in every bounded closed subset of D, then Theorem II implies that, except for a trivial special case, every isomorphism between $R(D)$ and $R(\Delta)$ is of necessity a homeomorphism.

To prove the theorems we consider a fixed isomorphism between $R(D)$ and $R(\Delta)$. It takes a function $f(z)$, $z \in D$, into a function $f^*(\zeta)$, $\zeta \in \Delta$, a set $S \subset R(D)$ into a set $S^* \subset R(\Delta)$. Let c be a complex constant.

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[1] After this paper was completed the author learned about a closely related unpublished result which was obtained by C. Chevalley and S. Kakutani several years ago. Chevalley and Kakutani proved that if to each boundary point W of B there exists a bounded analytic function defined in B and possessing at W a singularity then B is determined (modulo a conformal transformation) by the ring of all bounded analytic functions. The author is indebted to Professor Chevalley for the opportunity of reading a draft of the paper containing the proof.

For the sake of brevity we denote the element of $R(D)$ corresponding to the functions $f(z)\equiv c$ (or the element of $R(\Delta)$ corresponding to the function $g(\zeta)\equiv c$) by the letter c. We call a complex number rational if its real and imaginary parts are rational.

LEMMA 1. *Either $i^*=i$ and for every rational complex constant $r: r^*=r$, or $i^*=-i$ and $r^*=\bar{r}$.*

The proof is clear.

LEMMA 2. *If c is a constant, so is c^*.*

PROOF. If c is rational the assertion is contained in the preceding lemma. Irrational constants c are characterized by the existence of the inverse of the element $c-r$ for every rational constant r.

LEMMA 3. *All elements of $R(D)$ are constants if and only if D is the whole complex plane including the point at infinity.*

The proof is clear.

Lemmas 2 and 3 contain the proof of Theorem I for the case when D is the domain $0\leqq|z|\leqq\infty$. In what follows we consider only domains possessing boundary points. Without loss of generality we assume that neither D nor Δ contains the point at infinity. We also assume that $i^*=i$; the case $i^*=-i$ can be treated in the same way.

We denote the set of all functions belonging to $R(D)$ and vanishing at a point $a\in D$ by I_a. The set $I_\alpha\subset R(\Delta)$, $\alpha\in\Delta$, is defined similarly.

LEMMA 4. *There exists a one-to-one mapping $z\rightarrow z'=\phi(z)$ of D onto Δ such that $I_a^*=I_{\phi(a)}$.*

PROOF. Every element of $R(D)$ generates a principal ideal (f), that is, the set of all elements of the form fh, $h\in R(D)$. (f) is said to be a maximal principal ideal if $(f)\neq R(D)$ and if $(f)\subset(g)\neq R(D)$ implies that $(f)=(g)$. It is clear that $(f)^*=(f^*)$ and that (f^*) is a m. p. ideal if and only if (f) is. Hence Lemma 4 is an immediate consequence of the following lemma.

LEMMA 5. *(f) is a m.p. ideal if and only if $(f)=I_a$.*

PROOF. I_a is the principal ideal generated by the function $z-a$. If $I_a\subset(g)\neq R(D)$, then $g(z)$ must possess zeros in D. Since $z-a=g(z)h(z)$, $h\in R(D)$, $g(a)=0$ and $g\in I_a$. On the other hand, if $f(z)$ has no zeros in D, then $(f)=R(D)$, and if $f(a)=f'(a)=0$, or if $f(a)=f(b)=0$, $a\neq b$, then (f) is contained in and different from the principal ideal generated by the function $z-a$. It follows that (f) is a m.p. ideal if and only if $f(z)=(z-a)e^{h(z)}$, $a\in D$, $h\in R(D)$.

LEMMA 6. *For every point* $z_0\in D$, $f(z_0)^*=f^*[\phi(z_0)]$.

PROOF. If c is a constant such that $f(z_0)=c$, then $c-f$ belongs to I_{z_0}, so that c^*-f^* belongs to $I_{\phi(z_0)}$ and $f^*[\phi(z_0)]=c^*$.

The following two lemmas are immediate consequences of Lemma 6.

LEMMA 7. *If* $z_0\in D$ *and* $f(z_0)$ *is a rational number, then* $f(z_0)=f^*[\phi(z_0)]$.

LEMMA 8. *If* $f(z)$ *is univalent in* D, *then* $f^*(\zeta)$ *is univalent in* Δ.

LEMMA 9. *Let* $f(z)$ *be a univalent function defined in* D, *let* $f(D)$ *be the image of* D *under the transformation* $w=f(z)$, *and let* W *be the (finite) limit of a convergent sequence of distinct rational points* $\{w_n\}$ *belonging to* $f(D)$. W *is a boundary point of* $f(D)$ *if and only if there exists a function* $g(z)\in R(D)$ *such that* $g[h(w_n)]=n$, $h(w)$ *being the function inverse to* $w=f(z)$.

PROOF. If W is a boundary point of $f(D)$, choose an entire function $F(Z)$ such that $F[(W-w_n)^{-1}]=n$. The function $g(z)=F\{[W-f(z)]^{-1}\}$ satisfies the conditions of the lemma. On the other hand, if W is an interior point of $f(D)$, $W=f(a)$, $a\in D$, and for every $g(z)\in R(D)$, $\lim g[h(w_n)]$ exists and is finite.

LEMMA 10. *Let* $f(z)$ *be a univalent function defined in* D, *so that* $f^*(\zeta)$ *is a univalent function defined in* Δ. *The domains* $f(D)$ *and* $f^*(\Delta)$ *are identical.*

PROOF. It follows from Lemma 7 that the rational points belonging to $f(D)$ are identical with the rational points belonging to $f^*(\Delta)$. $f(D)$ is the set of all limit points of its rational points, except those limit points which lie on the boundary of $f(D)$. A similar remark applies to $f^*(\Delta)$. But Lemmas 7 and 9 imply that if a sequence of rational points from $f(D)$ converges to a boundary point W, W is a boundary point of $f^*(\Delta)$.

Lemma 10 contains the proof of Theorem I for domains possessing boundary points.

The proof of Theorem II depends on the following lemma.

LEMMA 11. *If* D *possesses boundary points, then* $c^*=c$ *for every constant* c.

We prove this lemma in several steps. Let B be any domain. By

$m[B]$ we denote the set of all complex numbers d such that the translation $Z=z+d$ maps B onto itself.

LEMMA 12. *For every univalent function* $f\in R(D)$ *and for every constant* c *the difference* $c-c^*$ *belongs to* $m[f(D)]$.

PROOF. The function $f_1=f+c$ is univalent in D, and the functions f^* and $f_1^*=f^*+c^*$ are univalent in Δ (Lemma 8). By virtue of Lemma 10, $f(D)$ is identical with $f^*(\Delta)$, and $f_1(D)$ is identical with $f_1^*(\Delta)$. But a translation by c takes $f(D)$ into $f_1(D)$ and a translation by $-c^*$ takes $f_1^*(\Delta)$ into $f^*(\Delta)$. Thus the translation by $c-c^*$ leaves $f(D)$ invariant.

LEMMA 13. *If* D *possesses finite boundary points, then there exists a univalent function* $f\in R(D)$ *such that* $m[f(D)]$ *is a discrete set.*

PROOF. Let f be a fixed univalent function defined in D. The set $m[f(D)]$ is closed and a modul, that is, it contains d_1-d_2 whenever it contains d_1 and d_2. Assume that $m[f(D)]$ is not discrete. Then it either contains all points, or all points of a straight line. If W is a finite boundary point of $f(D)$, every point $W+d$, $d\in m[f(D)]$, is a boundary point. It follows that the boundary of $f(D)$ contains a finite straight segment S. Let $Z(w)$ be the function which maps the domain exterior to S conformally onto $|Z|<1$. The function $g(z)=Z[f(z)]$ is univalent and bounded in D. It follows that $m[g(D)]$ contains only the point 0.

Now we can prove Lemma 11 under the hypothesis of Lemma 13. Let $f\in R(D)$ be univalent and such that $m[f(D)]$ is discrete. For every positive number t set $f_t=tf$. By Lemma 12 the difference $c-c^*$ belongs to $m[f_t(D)]$, that is, the number $(c-c^*)/t$ belongs to $m[f(D)]$. It follows that $c-c^*=0$.[2]

It remains to establish Lemma 11 for the case of the whole finite plane.

LEMMA 14. *If* D *is the domain* $|z|<\infty$ *and* Δ *the domain* $|\zeta|<\infty$, *then* $c_n\to\infty$ *implies* $c_n^*\to\infty$.

PROOF. Set $f(z)=z$. Then $f^*(\zeta)$ is univalent in Δ, that is, $f^*(\zeta)=A\zeta+B$, A, $B=$const., $A\neq 0$. For this f and for $z_0=c$, Lemma 6 yields $c^*=A\phi(c)+B$. Hence $c_n^*\to\infty$ whenever $\phi(c_n)\to\infty$. But $c_n\to\infty$

[2] An alternative argument was suggested to the author by C. Loewner. Assume that $c-c^*=re^{i\theta}$, $r\neq 0$. It is easy to see that for any univalent f and for any domain D satisfying the hypothesis of Lemma 13 there exists a (finite or semi-infinite) straight segment S whose interior points belong to $f(D)$ and whose end points are boundary points of $f(D)$. By a linear transformation we can achieve that S be the semi-infinite segment $z=te^{i\theta}$, $t>0$. It follows that $c-c^*$ does not belong to $m[f(D)]$.

implies the existence of a function $g \in R(D)$ such that $g(c_n)$ is rational and $g(c_n) \to \infty$. By Lemma 7, $g^*[\phi(c_n)] \to \infty$, from which it follows that $\phi(c_n) \to \infty$.

In order to prove Lemma 11 under the hypothesis of Lemma 14 we note that the transformation $c \to c^*$ is an automorphism of the complex field. This automorphism is continuous by virtue of Lemma 14. Hence $c^* = c$, for we assumed that $i^* = i$.

Lemma 11 being established, Lemma 6 yields the following lemma.

LEMMA 15. *If D possesses boundary points, then $f(z_0) = f^*[\phi(z_0)]$ for every $z_0 \in D$.*

This lemma would contain Theorem II if we would know that $\phi(z)$ is analytic in D. To show this we select for f the function $f(z) = z$. Lemma 15 shows that ϕ is the function inverse to the univalent (analytic) function f^*.

In the statement of Theorem I the ring of *all* analytic functions cannot be replaced by the subring $B(D)$ of all bounded analytic functions defined in D, even if $B(D)$ is treated as a normed ring (with $\|f\| = \text{l.u.b. } |f(z)|$), and the isomorphism between $B(D)$ and $B(\Delta)$ is required to be norm preserving.[3] In fact, let D be the domain $|z| < 1$, and let Δ be the domain $0 < |z| < 1$. The normed rings $B(D)$ and $B(\Delta)$ are identical.[4]

Neither is it possible to replace $R(D)$ by the linear space $L(D)$ of all analytic functions defined in D, even if $L(D)$ is considered to be a topological space (with the topology defined above) and the isomorphism between $L(D)$ and $L(\Delta)$ is required to be a homeomorphism.

In fact, let D be the domain $0 < r < |z| < 1$, and let D_1 and D_2 denote the domains $|z| < 1$ and $|z| > r$, respectively. Every function $f \in L(D)$ admits a unique Laurent decomposition: $f(z) = g(z) + h(z)/z$, $g \in L(D_1)$, $h \in L(D_2)$. It is easy to see that $f_n \to f$ if and only if $g_n \to g$ and $h_n \to h$. On the other hand, the linear subspace of $L(D_2)$ consisting of all functions of the form $h(z)/z$, $h \in L(D_2)$, is topologically isomorphic to $L(D_2)$, and $L(D_2)$ is topologically isomorphic to $L(D_1)$. It follows that $L(D)$ is topologically isomorphic to the direct sum of two spaces $L(D_1)$. Thus $L(D)$ considered as an abstract linear topological space is independent of r.

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[3] As a matter of fact, every isomorphism is.

[4] Cf., however, footnote 1.