SOME LIMIT THEOREMS

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1. **Introduction.** It is a classical result in the theory of trigonometric series that *if*

$$c_n \cos nx + d_n \sin nx \to 0 \qquad (n \to \infty) \tag{1.1}$$

for all (real) x on a set of positive measure, then

$$c_n \to 0, \qquad d_n \to 0. \tag{1.2}$$

Cantor proved this for the case that set $\{x\}$ is an interval, and Lebesgue established the result for a set of measure zero. A short proof is given by Hardy and Rogosinski.[1]

The following related result was proved and used by Szász.[2] *If*

$$a_n \sin nx + b_n \sin (n+1)x \to 0 \tag{1.3}$$

on a (real) set $\{x\}$ of positive measure, then

$$a_n \to 0, \qquad b_n \to 0. \tag{1.4}$$

Relations (1.1) and (1.3) can be put into complex form. For example, (1.1) becomes

$$a_n \exp \{nx\} + b_n \exp \{-nx\} \to 0, \tag{1.5}$$

with the conclusion that

$$a_n \to 0, \qquad b_n \to 0. \tag{1.6}$$

Here exp $\{u\}$ is defined by

$$\exp \{u\} \equiv e^{iu} \qquad (i = (-1)^{1/2}). \tag{1.7}$$

Our purpose in the present work is to extend the conclusions of the above-mentioned results to combinations more general than (1.3), (1.5). Thus in §2 we go from two terms to *k terms* and generalize the exponents; in §3 the coefficients of the exponentials are permitted

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[1] Hardy and Rogosinski, *Fourier series* (Cambridge Tracts in Mathematics and Mathematical Physics, no. 38), Theorem 92, p. 84.

[2] Otto Szász, *On Lebesgue summability and its generalization to integrals*, Amer. J. Math. vol. 67 (1945) pp. 389–396, especially Lemma 2, p. 395. Dr. Szász has informed me that, with the intention of using it in work on trigonometric series, he has proved (but not published) a generalization of (1.3), namely where the left side of (1.3) is replaced by the expression $\sum_{s=n}^{n+k} a_s e^{isx}$.

to be polynomials; and in §4 the multi-dimensional case is taken up.

2. **One-dimensional case.** In the following sections we suppose without further mention that all variables x, y, $\cdots$ are real.

LEMMA 2.1. *Let* $\{u_n\}$ *be a real sequence that does not have zero as its limit. The relation*

$$(2.1) \qquad \lim_{n\to\infty} \exp \{u_n x\} = 1$$

cannot hold on a set of positive measure.[3]

Suppose the lemma is false, so that there is a set $\mathcal{J}$ of positive measure on which (2.1) is satisfied. We may suppose that $\mathcal{J}$ is bounded. It is no restriction to assume that zero is not a limit point of $\{u_n\}$; for there exists an infinite subsequence of $\{u_n\}$ for which zero is not a limit point, and we may remove all u_n's not in this subsequence.

Suppose $\{u_n\}$ contains a bounded subsequence $\{u_{n_j}\}$; then from $\{u_{n_j}\}$ a further sequence can be chosen for which a limit exists. This limit, say L, cannot be zero, so from $\exp \{Lx\} = 1$ for x in $\mathcal{J}$ we conclude that $\mathcal{J}$ is at most a denumerable set, contrary to the assumption that $\mathcal{J}$ is of positive measure.

Now suppose that $\{u_n\}$ contains no bounded subsequence, so that $|u_n| \to \infty$. By Egoroff's theorem there is a subset $\mathcal{J}_1$ of $\mathcal{J}$, of positive measure, on which (2.1), that is,

$$\cos u_n x + i \sin u_n x \to 1,$$

holds uniformly. Consequently, since $1 + \cos u_n x$ is uniformly bounded,

$$\cos^2 u_n x - 1 \to 0 \qquad \text{(uniformly on } \mathcal{J}_1).$$

Integrating over $\mathcal{J}_1$:

$$(2.2) \qquad \int_{\mathcal{J}_1} \cos^2 u_n x dx \to \int_{\mathcal{J}_1} dx = m(\mathcal{J}_1).$$

Now there exists an open set $\mathfrak{Q}$, consisting of a finite number of nonoverlapping intervals, say (a_j, b_j), $j = 1, \cdots, p$, with the following two properties: (i) $\mathfrak{Q}$ contains $\mathcal{J}_1$; (ii) $m(\mathcal{J}_1) \leqq m(\mathfrak{Q}) < 3m(\mathcal{J}_1)/2$. Hence,

[3] As originally stated and proved, this and the other results of the paper used the condition *interval* everywhere in place of *set of positive measure*. We owe to Dr. Szász the suggestion to generalize to the case of positive measure, and also to extend the results to more than one variable (see §4).

$$\int_{\mathcal{J}_1} \cos^2 u_n x dx \leqq \int_{\mathcal{Q}} \cos^2 u_n x dx = 2^{-1} \int_{\mathcal{Q}} (1 + \cos 2u_n x) dx$$
$$= 2^{-1} m(\mathcal{Q}) + 2^{-1} \sum_{j=1}^{p} \left[\frac{\sin 2u_n x}{2u_n} \right]_{a_j}^{b_j};$$

so for all n sufficiently large,

$$\int_{\mathcal{J}_1} \cos^2 u_n x dx < \frac{3}{4} m(\mathcal{J}_1). \tag{2.3}$$

But this contradicts (2.2), so the lemma cannot be false.[4]

LEMMA 2.2. *Let* $\{t_{s,n}\}$, $s=1, \cdots, k$, *be real sequences with the property that none of the sequences* $\{t_{s,n}\}$, $\{t_{s,n}-t_{p,n}\}$ $(s\neq p)$ *has zero as a limit point. If real or complex constants* $\{A_{s,n}\}$ *exist such that*

$$\sum_{s=1}^{k} A_{s,n}[\exp \{t_{s,n} x\} - 1] \to 0 \tag{2.4}$$

for all x *on a set* $\mathcal{E}$ *of positive measure, then*

$$A_{s,n} \to 0 \qquad (s = 1, \cdots, k). \tag{2.5}$$

If $k=1$ the lemma is true in virtue of Lemma 2.1. Assume it true for the case $k-1$; we shall then prove it for k by an induction argument, and this will establish the truth of Lemma 2.2.

It is no restriction to suppose that $\mathcal{E}$ is a bounded set. $\mathcal{E}$ contains a point x_1 with the property that every interval containing x_1 in its interior meets $\mathcal{E}$ in a set of positive measure.[5] For suppose not. Then about each x in $\mathcal{E}$ exists an interval I_x, with x in its interior, such that $\mathcal{E} \cdot I_x$ is of measure zero. Let x_1 be in $\mathcal{E}$, and let I_{x_1} be the largest associated interval. It is clear that there *is* a largest interval. If x_2 in $\mathcal{E}$ is not in I_{x_1}, then it too has a largest associated interval I_{x_2}, and I_{x_1}, I_{x_2} do not meet. It is now a straightforward argument to show that $\mathcal{E}$ is covered by at most a denumerable number of such intervals I_x, thus establishing $\mathcal{E}$ as a set of zero measure. This contradiction shows that a point such as the aforementioned x_1 exists.

Let x in (2.4) take on such a value x_1 and subtract from (2.4). There results the relation

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[4] This contradiction does not preclude the possibility that the set of points for which (2.1) holds is non-measurable, but in this case the set cannot contain a subset of positive measure.

[5] A much stronger conclusion as to the number of such points x_1 is of course possible, but we require only the above mild assertion.

$$\sum_{s=1}^{k} A_{s,n} \exp \{t_{s,n}x_1\} [\exp \{t_{s,n}y\} - 1] \to 0 \tag{2.6}$$

for all y in $\mathcal{E}_1 \equiv \{y = x - x_1$, as x ranges over $\mathcal{E}\}$. $\mathcal{E}_1$ and $\mathcal{E}$ have the same (positive) measure, and the replacement of (2.4) and $\mathcal{E}$ by (2.6) and $\mathcal{E}_1$ insures that the origin ($y = 0$) is a point of $\mathcal{E}_1$ every neighborhood of which contains a subset of $\mathcal{E}_1$ of positive measure.

Let

$$B_{s,n} = A_{s,n} \exp \{t_{s,n}x_1\}. \tag{2.7}$$

Then

$$\sum_{s=1}^{k} B_{s,n}[\exp \{t_{s,n}y\} - 1] \to 0 \qquad (y \text{ in } \mathcal{E}_1), \tag{2.8}$$

and (2.5) is equivalent to

$$B_{s,n} \to 0 \qquad (s = 1, \cdots, k). \tag{2.9}$$

Suppose the lemma is false for case k. Then a value s, say $s = k$, exists such that[6] $B \to 0$ is false. There therefore is a subsequence $\{n_j \equiv n(j)\}$ of $\{n\}$, and a number $M > 0$, such that

$$|B_{k,n(j)}| > M;$$

so on replacing $\{n\}$ by $\{n(j)\}$ in (2.8) and dividing by $B_{k,n(j)}$, we have

$$[\exp \{t_{k,n(j)}y\} - 1] + \sum_{s=1}^{k-1} C_{s,n(j)}[\exp \{t_{s,n(j)}y\} - 1] \to 0 \qquad (y \text{ in } \mathcal{E}_1). \tag{2.10}$$

Here

$$C_{s,n(j)} = \frac{B_{s,n(j)}}{B_{k,n(j)}}. \tag{2.11}$$

Let y_1 be an arbitrary point of $\mathcal{E}_1$. The set of points $\{u\}$ defined by $u = y - y_1$ as y ranges over $\mathcal{E}_1$ will be denoted by $\mathcal{E}_{y_1}$, and will be termed a *translation set* (relative to $\mathcal{E}_1$). Clearly, $m(\mathcal{E}_{y_1}) = m(\mathcal{E}_1)$.

Let $\mathfrak{Q}$ be an open set containing $\mathcal{E}_1$, with $m(\mathcal{E}_1) < m(\mathfrak{Q}) < 3m(\mathcal{E}_1)/2$. There exists a set of nonoverlapping open intervals $\mathfrak{Q}_1 = I_1 + \cdots + I_r$ contained in $\mathfrak{Q}$, for which

[6] Actually, for *every* $s = 1, \cdots, k$ the quantity $B_{s,n}$ does not approach zero; for otherwise we can drop from sum (2.8) all terms for which $B_{s,n} \to 0$, and thus reduce (2.8) to $k-1$ or less terms, in which case the lemma is true by our induction assumption.

$$m(\mathcal{E}_1) < m(\mathcal{Q}_1) < 3m(\mathcal{E}_1)/2,$$

and also such that

$$m(\mathcal{Q}_1 \cdot \mathcal{E}_1) > 3m(\mathcal{E}_1)/4.$$

To each end of I_p ($p=1, \cdots, r$) add an interval of length Δ, forming a new interval J_p, where Δ is chosen small enough so that on setting $\mathcal{Q}_2 = J_1 + \cdots + J_q$ ($q \leqq r$ since some intervals may overlap and thus be combined) then $m(\mathcal{Q}_2) < 3m(\mathcal{E}_1)/2$.

Let $\mathcal{H}$ be the subset of numbers y of $\mathcal{E}_1$ for which $|y| < \Delta$. We know that $\mathcal{H}$ is of positive measure. Moreover, for an arbitrary y in $\mathcal{H}$,

$$m(\mathcal{Q}_2 \cdot \mathcal{E}_y) > 3m(\mathcal{E}_1)/4, \qquad m(\mathcal{Q}_2 \cdot \mathcal{E}_1) > 3m(\mathcal{E}_1)/4.$$

Since $m(\mathcal{Q}_2) < 3m(\mathcal{E}_1)/2$, it follows that

$$m(\mathcal{E}_1 \cdot \mathcal{E}_y) > 0 \qquad \text{(all } y \text{ in } \mathcal{H}\text{)}.$$

We see from (2.10) and Lemma 2.1 that we cannot have $C_{s,n(j)} \to 0$ for all $s=1, \cdots, k-1$. Hence there is an s, say $s=1$, for which $C_{1,n(j)} \to 0$ is false; and a subsequence $\{m(j)\}$ of $\{n(j)\}$, and a positive number K, such that

$$|C_{1,m(j)}| > K. \tag{2.12}$$

Now the relation

$$\exp\{(t_{1,m(j)} - t_{k,m(j)})z\} \to 1 \qquad (j \to \infty) \tag{2.13}$$

cannot hold on a set of positive measure (Lemma 2.1); consequently, there is a point y_1 in $\mathcal{H}$ such that (2.13) is false for $z = y_1$. Choose $y = y_1$ in (2.10) and subtract from (2.10):

$$\begin{aligned} &\exp\{t_{k,n(j)}y_1\}[\exp\{t_{k,n(j)}u\} - 1] \\ &\qquad + \sum_{s=1}^{k-1} C_{s,n(j)} \exp\{t_{s,n(j)}y_1\}[\exp\{t_{s,n(j)}u\} - 1] \to 0, \end{aligned} \tag{2.14}$$

where $u = y - y_1$ (y in $\mathcal{E}_1$), so that u ranges over the translation set $\mathcal{E}_{y_1}$. In (2.14), replace $\{n(j)\}$ by $\{m(j)\}$ and divide by $\exp\{t_{k,m(j)}\, y_1\}$. This gives us

$$\begin{aligned} &[\exp\{t_{k,m(j)}u\} - 1] \\ &+ \sum_{s=1}^{k-1} C_{s,m(j)} \exp\{(t_{s,m(j)} - t_{k,m(j)})y_1\}[\exp\{t_{s,m(j)}u\} - 1] \to 0 \\ &\qquad\qquad\qquad (u \text{ in } \mathcal{E}_{y_1}). \end{aligned} \tag{2.15}$$

Let $\mathcal{L} = \mathcal{E}_1 \cdot \mathcal{E}_{y_1}$. We know that $m(\mathcal{L}) > 0$. If we restrict u in (2.15) and y in (2.10) to lie in $\mathcal{L}$, then u and y may be identified; so on subtracting (2.15) from (2.10) (with $n(j)$ replaced by $m(j)$) we obtain

$$(2.16)\quad \sum_{s=1}^{k-1} C_{s,m(j)}[1-\exp\{(t_{s,m(j)}-t_{k,m(j)})y_1\}][\exp\{t_{s,m(j)}y\}-1]\to 0 \qquad (y \text{ in } \mathcal{L}).$$

This relation has only $k-1$ terms, and for it the hypotheses of Lemma 2.2 hold. By our induction assumption, therefore, each coefficient approaches zero. Since (2.12) holds, we must have

$$\exp\{(t_{1,m(j)} - t_{k,m(j)})y_1\} \to 1,$$

which is contrary to the choice of y_1.

Thus the induction chain is complete, and the lemma is established.

THEOREM 2.1. *Let* $\{a_{s,n}\}$, $s=1, \cdots, k$, *be real or complex number sequences, and let the real sequences* $\{r_{s,n}\}$ *have the property that none of the sequences* $\{r_{s,n}-r_{p,n}\}$ $(s\neq p)$ *has zero as a limit point. If*

$$(2.17)\qquad \sum_{s=1}^{k} a_{s,n} \exp\{r_{s,n}x\} \to 0$$

for all x *on a set* $\mathcal{E}$ *of positive measure, then*

$$(2.18)\qquad a_{s,n} \to 0 \qquad (s = 1, \cdots, k).$$

REMARK. For sequences $\{r_{s,n}\}$ satisfying the above hypothesis, Theorem 2.1 asserts what may be termed the *asymptotic linear independence* of the functions $\exp\{r_{s,n}x\}$, $s=1, \cdots, k$.

If the theorem is false, there is an index s, say $s=1$, for which $a_{1,n}\to 0$ is false; so a subsequence $\{n(j)\}$ of $\{n\}$ exists, and a positive number M, such that $|a_{1,n(j)}|>M$. Replace $\{n\}$ by $\{n(j)\}$ in (2.17) and divide by $a_{1,n(j)} \exp\{r_{1,n(j)}x\}$:

$$(2.19)\qquad 1 + \sum_{s=2}^{k} b_{s,n(j)} \exp\{(r_{s,n(j)} - r_{1,n(j)})x\} \to 0,$$

where

$$(2.20)\qquad b_{s,n(j)} = \frac{a_{s,n(j)}}{a_{1,n(j)}}.$$

Take $x=x_1$ in (2.19) and subtract from (2.19):

$$(2.21)\qquad \sum_{s=2}^{k} b_{s,n(j)} \exp\{t_{s,n(j)}x_1\}[\exp\{t_{s,n(j)}y\} - 1] \to 0.$$

Here

$$t_{s,n(j)} = r_{s,n(j)} - r_{1,n(j)}; \quad y = x - x_1 \qquad (x \text{ in } \mathcal{E}),$$

so y ranges over a set $\mathcal{E}_1$ of positive measure.

The hypotheses of Lemma 2.2 are satisfied, so

$$b_{s,n(j)} \to 0 \qquad (s = 2, \cdots, k).$$

But this contradicts (2.19). Thus Theorem 2.1 is established.

COROLLARY 2.1. *Let the sequences* $\{t_{s,n}\}$ *satisfy the hypothesis of Lemma* 2.2. *If a constant* A *and constants* $\{a_{s,n}\}$ *exist such that*

$$\sum_{s=1}^{k} a_{s,n} \exp \{t_{s,n}x\} \to A \tag{2.22}$$

for all x *on a set of positive measure, then*

$$A = 0; \quad a_{s,n} \to 0 \qquad (s = 1, \cdots, k). \tag{2.23}$$

For, (2.22) can be written

$$\sum_{s=1}^{k+1} a_{s,n} \exp \{t_{s,n}x\} \to 0, \tag{2.24}$$

where

$$a_{k+1,n} \equiv -A, \qquad t_{k+1,n} \equiv 0. \tag{2.25}$$

The hypothesis of Theorem 2.1 is fulfilled in (2.24), so $a_{s,n} \to 0$, $s = 1, \cdots, k+1$.

In Theorem 2.1 the condition on the sequences $\{r_{s,n}\}$ cannot be weakened. This is shown by the following theorem.

THEOREM 2.2. *Let the real sequences* $\{r_{s,n}\}$, $s = 1, \cdots, k$, *be such that at least one of the sequences* $\{r_{s,n} - r_{p,n}\}$ $(s \neq p)$ *has zero as a limit point. There exist sequences* $\{a_{s,n}\}$, *at least one of which does not approach zero, such that* (2.17) *holds for all* x.

We may suppose that $t_{n(j)} \equiv r_{1,n(j)} - r_{2,n(j)} \to 0$ as $j \to \infty$. Choose $a_{s,n} = 0$, $s = 3, \cdots, k$, and $a_{1,n} = a_{2,n} = 0$ for $n \neq n_1, n_2, \cdots$. The left side of (2.17) becomes

$$a_{1,n(j)} \exp \{r_{1,n(j)}x\} + a_{2,n(j)} \exp \{r_{2,n(j)}x\}, \tag{2.26}$$

and this approaches zero if and only if

$$a_{1,n(j)} \exp \{t_{n(j)}x\} + a_{2,n(j)} \to 0. \tag{2.27}$$

It is clear that if we *define* $a_{1,n(j)} = 1$, $a_{2,n(j)} = -1$, then (2.27) does hold for every x. Actually, these coefficients can be chosen to be *unbounded*. For let $R>0$ be given. There exists an $M=M(R)$ such that

$$\left| \exp\left\{t_{n(j)}x\right\} - 1\right| \leqq M \left| t_{n(j)}\right|$$

for all $|x| \leqq R$. On writing the left side of (2.27) as

$$a_{1,n(j)}\left[\exp\left\{t_{n(j)}x\right\} - 1\right] + \left[a_{1,n(j)} + a_{2,n(j)}\right],$$

we see that (2.27) follows if we choose $a_{1,n(j)}$, $a_{2,n(j)}$ so that

$$a_{1,n(j)}t_{n(j)} \to 0, \qquad a_{1,n(j)} + a_{2,n(j)} \to 0. \tag{2.28}$$

Since $t_{n(j)} \to 0$, conditions (2.28) can be satisfied by sequences $a_{1,n(j)}$, $a_{2,n(j)}$ that are unbounded.

3. **Polynomial coefficients.** The result of Theorem 2.1 can be extended to the case of polynomial coefficients of bounded degree:

THEOREM 3.1. *Let $\{r_{s,n}\}$, $s=1, \cdots, k$, be real sequences such that none of the sequences $\{r_{s,n}-r_{p,n}\}$ $(s\neq p)$ has zero as a limit point. Let $\{P_{s,n}(x)\}$ be real or complex polynomial sequences:*

$$P_{s,n}(x) = a_{s,0,n} + a_{s,1,n}x + \cdots + a_{s,q_s,n}x^{q_s} \qquad (s = 1, \cdots, k) \tag{3.1}$$

in which q_s is independent of n. If

$$\sum_{s=1}^{k} P_{s,n}(x) \exp\left\{r_{s,n}x\right\} \to 0 \tag{3.2}$$

for all x on a set $\mathcal{E}$ of positive measure, then

$$a_{s,p,n} \to 0 \qquad (p = 0, 1, \cdots, q_s;\ s = 1, \cdots, k). \tag{3.3}$$

Let

$$q = \max\left\{q_1, \cdots, q_k\right\}. \tag{3.4}$$

If $q=0$ the result follows from Theorem 2.1. Suppose the theorem is false. Then there is an integer $Q>0$ such that whenever $q<Q$ the result is true, but for at least one case with $q=Q$ the theorem is untrue. In each case of failure, with $q=Q$, at least one polynomial coefficient is of degree Q. Let λ be the number of such polynomials; then there is a positive integer Λ with the property that whenever $\lambda<\Lambda$ (and $q=Q$), the theorem is true, but there is a case $\lambda=\Lambda$, $q=Q$ for which it is false.

Let (3.2) be such a case, so that exactly Λ polynomials, that we may take to be $P_{s,n}(x)$, $s=1, \cdots, \Lambda$, are of degree Q while all other polynomial coefficients (if any) are of lower degree. Since the theorem is false for this case, not all the coefficients approach zero as $n\to\infty$. For each $n=1, 2, \cdots$ let

$$(3.5) \qquad \mu_n = \max \left\{ \left| a_{s,p,n} \right| \right\} \qquad (p = 0, 1, \cdots, q_s;\ s = 1, \cdots, k).$$

Then μ_n does not approach 0. There therefore exist values $s=\sigma$, $p=p_\sigma$, a positive number M, and a subsequence $\{n(j)\}$ of $\{n\}$, such that

$$(3.6) \qquad \left| a_{\sigma,p_\sigma,n(j)} \right| = \mu_{n(j)} > M \qquad (j = 1, 2, \cdots).$$

Replace $\{n\}$ by $\{n(j)\}$ in (3.2) and divide by $a_{\sigma,p_\sigma,n(j)} \exp \{r_{1,n(j)}x\}$:

$$(3.7) \qquad R_{1,n(j)}(x) + \sum_{s=2}^{k} R_{s,n(j)}(x) \exp \left\{ t_{s,n(j)}x \right\} \to 0 \qquad (x \text{ in } \mathcal{E}),$$

where

$$t_{s,n(j)} = r_{s,n(j)} - r_{1,n(j)}$$

and

$$R_{s,n(j)}(x) = \sum_{p=0}^{q_s} b_{s,p,n(j)}x^p \equiv \frac{1}{a_{\sigma,p_\sigma,n(j)}} \cdot P_{s,n(j)}(x).$$

The b-coefficients are bounded, and $b_{\sigma,r_\sigma,n(j)} \equiv 1$ for all j. Consequently, there exists a subsequence $\{m(j)\}$ of $\{n(j)\}$ for which the following limits exist:

$$(3.8) \qquad \lim_{j\to\infty} b_{s,p,m(j)} = b_{s,p} \qquad (p = 0, 1, \cdots, q_s;\ s = 1, \cdots, k);$$

and not all of $b_{s,p}$ are zero, since $b_{\sigma,p_\sigma}=1$.

From (3.7) it follows that

$$(3.9) \qquad R_1(x) + \sum_{s=2}^{k} R_s(x) \exp \left\{ t_{s\ m(j)}x \right\} \to 0 \qquad (x \text{ in } \mathcal{E}),$$

where

$$(3.10) \qquad R_s(x) = \sum_{p=0}^{q_s} b_{s,p}x^p \qquad (s = 1, \cdots, k).$$

Moreover, $R_1(x)$ is of degree Q; for if it is of lower degree, then (3.9) presents a case in which fewer than Λ polynomials are of degeee Q, so from the definition of Λ it will follow that the theorem is true for (3.9). Thus all coefficients in all the polynomials approach zero as $j\to\infty$. But this is contrary to the fact that $b_{\sigma,p_\sigma}=1$. Hence the degree of $R_1(x)$ must be Q.

We know from the proof of Lemma 2.2 that set $\mathcal{E}$ contains a point

x_1 every neighborhood of which contains a subset of $\mathcal{E}$ of positive measure; and using this fact, we may conclude (as was similarly argued in establishing Lemma 2.2) that there exist distinct numbers h_1, h_2 such that on defining $\mathcal{E}_1$, $\mathcal{E}_2$ by

$$\mathcal{E}_p \equiv \{y_p = x + h_p, \ x \text{ ranging over } \mathcal{E}\} \qquad (p = 1, 2),$$

then $\mathcal{E}_3 \equiv \mathcal{E}_1 \cdot \mathcal{E}_2$ is a set of positive measure.

Relation (3.9) may then be written in each of the forms

$$\begin{aligned} &R_1(y_p - h_p) \\ (3.11) \qquad &+ \sum_{s=2}^{k} R_s(y_p - h_p) \exp\{-t_{s,m(j)}h_p\} \exp\{t_{s,m(j)}y_p\} \to 0 \\ &\qquad\qquad (y_p \text{ in } \mathcal{E}_p, \ p = 1, 2). \end{aligned}$$

If we consider only points in $\mathcal{E}_3$, then y_1 and y_2 may be identified:

$$\begin{aligned} &R_1(y - h_p) \\ (3.12) \qquad &+ \sum_{s=2}^{k} R_s(y - h_p) \exp\{-t_{s,m(j)}h_p\} \exp\{t_{s,m(j)}y\} \to 0 \\ &\qquad\qquad (y \text{ in } \mathcal{E}_3, \ p = 1, 2). \end{aligned}$$

On subtracting we have

$$\begin{aligned} (3.13) \quad &[R_1(y - h_1) - R_1(y - h_2)] + \sum_{s=2}^{k} [R_s(y - h_1) \exp\{-t_{s,m(j)}h_1\} \\ &\qquad - R_s(y - h_2) \exp\{-t_{s,m(j)}h_2\}] \exp\{t_{s,m(j)}y\} \to 0 \\ &\qquad\qquad (y \text{ in } \mathcal{E}_3). \end{aligned}$$

Since $R_1(x)$ is of actual degree Q, and $Q>0$, we see that

$$(3.14) \qquad H(y) \equiv [R_1(y - h_1) - R_1(y - h_2)]$$

is a polynomial of degree exactly $Q-1$, and is therefore not identically zero. But $H(y)$ being of degree less than Q, this places (3.13) in the category of cases for which the theorem is true, since now fewer than Λ polynomials are of degree Q. Hence all coefficients approach zero. This is however contrary to the condition that $H(y) \not\equiv 0$.

We have thus arrived at a contradiction, so the assumption that Theorem 3.1 is false is untenable.

4. **Higher dimensions.** We shall now show that the foregoing results extend to the general case of p dimensions. Throughout this section the term *measure* refers to *p-dimensional measure*. Proofs for

the general case usually follow those of the preceding sections, and are accordingly given briefly or not at all.

LEMMA 4.1. *Let $\{u_{s,n}\}$, $s=1, \cdots, p$, be real sequences such that for at least one value of s, $\{u_{s,n}\}$ does not have zero as its limit. The relation*

$$\lim_{n\to\infty} \exp\left\{\sum_{s=1}^{p} u_{s,n}x_s\right\} = 1 \tag{4.1}$$

cannot hold on a set of points $(x)\equiv(x_1, \cdots, x_p)$ of positive measure.

Assume that the lemma is false, so there is a set $\mathcal{J}$ of positive measure for which (4.1) holds. Suppose $s=q$ is the value for which $u_{q,n}$ does not approach 0. We may then assume that $\{u_{q,n}\}$ does not have zero as limit point. If a subsequence $\{n(j)\}$ of $\{n\}$ exists for which all the sequences $\{u_{s,n(j)}\}$, $s=1, \cdots, p$, are bounded, then there will be a further subsequence $\{m(j)\}$ for which the following limits exist:

$$\lim_{j\to\infty} u_{s,m(j)} = l_s \qquad (s = 1, \cdots, p),$$

with $l_q\neq 0$. Hence if (x) is in $\mathcal{J}$, then (x) must satisfy one of the equations

$$\frac{1}{2\pi}\sum_{s=1}^{p} l_s x_s = 0, \ \pm 1, \ \pm 2, \cdots. \tag{4.2}$$

For each choice of the right side, (4.2) is a hyperplane, and is of measure zero. The totality of planes (4.2) is likewise of zero measure, and so, therefore, is $\mathcal{J}$, which is contrary to assumption.

There remains to consider the case where for at least one value of s, say $s=1$, and a subsequence $\{n(j)\}$,

$$| u_{1,n(j)} | \to \infty.$$

The remainder of the argument now follows that of Lemma 2.1, with obvious p-dimensional modifications.

LEMMA 4.2. *Let $\{t_{s,r,n}\}$, $s=1, \cdots, k$; $r=1, \cdots, p$ be real sequences with the following property: a value $r=q$ exists such that none of the sequences $\{t_{s,q,n}\}$, $\{t_{s,q,n}-t_{\sigma,q,n}\}$ $(s\neq\sigma)$ has zero as a limit point. If real or complex constants $\{A_{s,n}\}$ exist such that*

$$\sum_{s=1}^{k} A_{s,n}\left[\exp\left\{\sum_{r=1}^{p} t_{s,r,n}x_r\right\} - 1\right]\to 0 \tag{4.3}$$

for all $(x)\equiv(x_1, \cdots, x_p)$ on a set $\mathcal{E}$ of positive measure, then

(4.4) $$A_{s,n} \to 0 \qquad (s = 1, \cdots, k).$$

The proof is like that of Lemma 2.2 with simple modifications that need not be detailed here.

Lemma 4.2 leads directly to the following theorem.

THEOREM 4.1. *Let* $\{a_{s,n}\}$, $s=1, \cdots, k$, *be real or complex sequences, and let the real sequences* $\{q_{s,r,n}\}$, $s=1, \cdots, k$; $r=1, \cdots, p$, *be such that for some value* $r=\omega$, *none of the sequences* $\{q_{s,\omega,n}-q_{\sigma,\omega,n}\}$ $(s\neq\sigma)$ *has zero as a limit point. If*

(4.5) $$\sum_{s=1}^{k} a_{s,n} \exp\left\{\sum_{r=1}^{p} q_{s,r,n}x_r\right\} \to 0$$

for all $(x)\equiv(x_1, \cdots, x_p)$ *on a set* $\mathcal{E}$ *of positive measure, then*

(4.6) $$a_{s,n} \to 0 \qquad (s = 1, \cdots, k).$$

The proof follows an earlier one (Theorem 2.1), as does the next result:

COROLLARY 4.1. *Let the sequences* $\{t_{s,r,n}\}$ *satisfy the hypothesis of Lemma* 4.2. *If a constant* A *and constants* $\{a_{s,n}\}$ *exist such that*

(4.7) $$\sum_{s=1}^{k} a_{s,n} \exp\left\{\sum_{r=1}^{p} t_{s,r,n}x_r\right\} \to A$$

for all (x) *on a set of positive measure, then*

(4.8) $$A = 0; \qquad a_{s,n} \to 0 \qquad (s = 1, \cdots, k).$$

Finally, we have

THEOREM 4.2. *Let* $\{q_{s,r,n}\}$, $s=1, \cdots, k$; $r=1, \cdots, p$, *be real sequences satisfying the hypothesis of Theorem* 4.1. *Let* $\{P_{s,n}(x_1, \cdots, x_p)\}$ *be real or complex polynomial sequences*:

(4.9) $$P_{s,n}(x_1, \cdots, x_p) = \sum_{h_1+\cdots+h_p=0}^{e_s} a_{s,n;h_1,\cdots,h_p} x_1^{h_1} \cdots x_p^{h_p} \qquad (s = 1, \cdots, k),$$

in which e_s *is independent of* n. *If*

(4.10) $$\sum_{s=1}^{k} P_{s,n}(x_1, \cdots, x_p) \exp\left\{\sum_{r=1}^{p} q_{s,r,n}x_r\right\} \to 0$$

for all (x) *on a set* $\mathcal{E}$ *of positive measure, then*

$$(4.11) \qquad a_{s,n;h_1,\cdots,h_p} \to 0 \qquad (0 \leqq h_1 + \cdots + h_p \leqq e_s;\ s = 1, \cdots, k).$$

Up to a point the proof is patterned after that of Theorem 3.1. When the equivalent of (3.13) is obtained, however, we can no longer assert that $H(y_1, \cdots, y_p)$ is not identically zero simply from the fact that two distinct sets $(h)_1 = (h_{11}, \cdots, h_{1p})$, $(h)_2 = (h_{21}, \cdots, h_{2p})$ exist such that

$$(4.12) \qquad \begin{aligned} H(y_1, \cdots, y_p) \equiv R_1(y_1 - h_{11}, \cdots, y_p - h_{1p}) \\ - R_1(y_1 - h_{21}, \cdots, y_p - h_{2p}). \end{aligned}$$

In fact, nonconstant polynomials in more than one variable exist that are "periodic." We avoid this difficulty by observing that for a fixed point $(h)_1$, the point $(h)_2$ can be chosen arbitrarily on a set of positive measure. Examination of the proof of Lemma 2.2 shows this. Now if a polynomial $L(x_1, \cdots, x_p)$ has the property that

$$L(x_1 + c_1, \cdots, x_p + c_p) \equiv L(x_1, \cdots, x_p)$$

for all sets $(c) \equiv (c_1, \cdots, c_p)$ on a set of positive measure, then surely $L \equiv$constant.

In our case, therefore, if $H(y_1, \cdots, y_p) \equiv 0$ for all possible choices of $(h)_2$, then R_1 is a constant, contrary to the fact that its degree is $Q > 0$ (cf. Theorem 3.1). The remainder of the proof offers no difficulty.

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