THE NON-EXISTENCE OF A CERTAIN TYPE OF ODD PERFECT NUMBER

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For any perfect number[1] expressed in the form $n = a_0 a_1 \cdots a_t$, where

$$a_0 = p_0^{\alpha_0}, \; a_1 = p_1^{\alpha_1}, \cdots, a_t = p_t^{\alpha_t}$$

and $p_0, p_1, \cdots, p_t$ are the distinct prime factors of n, it can be shown that a unique one of the prime powers a_i has an even divisor sum $\sigma(a_i)$. Throughout we shall suppose that the primes p_i and hence the prime powers a_i to be so numbered that

$$(1) \qquad \sigma(a_0) \equiv 0; \quad \sigma(a_i) \equiv 1 \qquad i = 1, 2, \cdots, t, \pmod 2.$$

Then with the abbreviations

$$(2) \qquad \sigma_0 = \sigma(a_0)/2; \quad \sigma_i = \sigma(a_i), \qquad i = 1, 2, \cdots, t,$$

the condition for n to be perfect may be written in the form

$$(3) \qquad \sigma(n)/2 = \sigma_0 \sigma_1 \cdots \sigma_t = a_0 a_1 \cdots a_t = n.$$

For the even perfect numbers, which are the only kind known, it is well known that $p_0 = 2^q - 1$, $\alpha_0 = 1$, $p_1 = 2$, $\alpha_1 = q - 1$, $t = 1$, where q is any prime such that $2^q - 1$ is also prime. Then $\sigma_1 = 2^q - 1 = a_0$ and $\sigma_0 = 2^{q-1} = a_1$ so that σ_0 and σ_1 are the prime powers a_0 and a_1 in re-reverse order. It is natural to inquire whether there may exist odd perfect numbers such that analogously $\sigma_0, \sigma_1, \cdots, \sigma_t$ are the prime powers $a_0, a_1, \cdots, a_t$ in a different order. In the following it will be proved that no odd perfect numbers of this form can exist.

We first establish an algebraic identity. Throughout this paper the product notation $\prod_{i=a}^{b} x_i$ is used with the convention that $\prod_{i=a}^{b} x_i = 1$ if $a > b$.

LEMMA 1. *Let $c_1, c_2, \cdots, c_t$ be any $t \geqq 2$ integers (more generally, elements of a commutative ring with a unit element). Then,*

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[1] For a summary of results concerning perfect numbers (including those cited above) with references see L. Dickson, *History of the theory of numbers*, vol. 1, 1919, pp. 1–33. For a more recent paper with references to other recent literature on the subject, see A. Brauer, *On the non-existence of odd perfect numbers of form $p^\alpha q_1^2 q_2^2 \cdots q_{t-1}^2 q_t^4$*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 712–718.

$$\sum_{j=1}^{t}\left[\prod_{i=1}^{j-1}(c_i-1)\prod_{i=j+1}^{t}c_i\right]=\prod_{i=1}^{t}c_i-\prod_{i=1}^{t}(c_i-1). \tag{4}$$

Proof. The identity holds for $t=2$, both members reducing to c_1+c_2-1. Proceeding by induction, assume the identity holds for $t=m$. Multiplying both members by c_{m+1} and adding $\prod_{i=1}^{m}(c_i-1)$, we have

$$\begin{aligned}\sum_{j=1}^{m+1}\left[\prod_{i=1}^{j-1}(c_i-1)\prod_{i=j+1}^{m+1}c_i\right]\\ &= c_{m+1}\sum_{j=1}^{m}\left[\prod_{i=1}^{j-1}(c_i-1)\prod_{i=j+1}^{m}c_i\right]+\prod_{i=1}^{m}(c_i-1)\\ &= c_{m+1}\left[\prod_{i=1}^{m}c_i-\prod_{i=1}^{m}(c_i-1)\right]+\prod_{i=1}^{m}(c_i-1)\\ &= \prod_{i=1}^{m+1}c_i-\prod_{i=1}^{m+1}(c_i-1),\end{aligned}$$

the first and last members of which are the members of the required identity for $t=m+1$, thus completing the induction.

For an odd integer $n=a_0a_1\cdots a_t$ to be perfect a well known necessary condition is that with the p's numbered according to (1)

$$\alpha_0\equiv p_0\equiv 1 \pmod 4. \tag{5}$$

For such prime powers we have the following:

Lemma 2. *Let $\alpha>1$ be an integer and p a prime such that $\alpha\equiv p\equiv 1$ (mod 4). Then $\sigma(p^\alpha)$ is divisible by at least two distinct odd primes.*

Proof. It is sufficient to exhibit two odd nontrivial divisors of $\sigma(p^\alpha)$ which are relatively prime. We have

$$\sigma(p^\alpha)=\frac{p^{\alpha+1}-1}{p-1}=2\cdot\frac{p^{(\alpha+1)/2}+1}{2}\cdot\frac{p^{(\alpha+1)/2}-1}{p-1}\cdot \tag{6}$$

Then the required divisors are

$$d_1=\frac{p^{(\alpha+1)/2}+1}{2}\quad\text{and}\quad d_2=\frac{p^{(\alpha+1)/2}-1}{p-1}\cdot \tag{7}$$

They are both odd since $\sigma(p^\alpha)=1+p+p^2+\cdots+p^\alpha\equiv\alpha+1\equiv 2$ (mod 4). They are coprime, since $2d_1-(p-1)d_2=2$ so that if there were a common divisor of d_1 and d_2, it would have to divide 2. Finally, they are nontrivial divisors since $d_1>d_2>1$ for $\alpha>1$.

We are now able to prove our theorem.

THEOREM. *Let* $n = a_0 a_1 \cdots a_t$, *where*

$$a_0 = p_0^{\alpha_0},\ a_1 = p_1^{\alpha_1}, \cdots, a_t = p_t^{\alpha_t}$$

and $p_0, p_1, \cdots, p_t$ *are distinct odd primes. Then, if each of the quantities* σ_i, $i = 0, 1, \cdots, t$, *defined in* (2) *is a power of a prime, n is not a perfect number.*

PROOF. Assume that n is perfect. Then (3) holds; and since the σ_i are prime powers, by the fundamental theorem of arithmetic, they must each equal one of the a_j, $i, j = 0, \cdots, t$ with $i \neq j$, because $\sigma_i \equiv 1 \not\equiv 0 \pmod{p_i}$. That is, $\sigma_0, \cdots, \sigma_t$ are the prime powers $a_0, \cdots, a_t$ in a different order.

Without loss of generality we may suppose that the p's are numbered recursively in the following manner: p_0 has already been chosen in accord with (1) (or (5), which amounts to the same thing). Choose as p_1 that prime p_i for which $a_i = \sigma_0$, as p_2 that prime p_j for which $a_j = \sigma_1$, and in general choose as p_m that prime p_r for which $a_r = \sigma_{m-1}$. This process can be continued until a prime p_k is reached such that $\sigma_k = a_l$ with $l < k$ so that we cannot set $a_{k+1} = \sigma_k$. We shall now show that this cannot occur until the primes have been completely numbered; that is, when $k = t$, and then $l = 0$. First suppose $0 < l < k \leqq t$. Then we have both $\sigma_k = a_l$ and $\sigma_{l-1} = a_l$ so that in the product, $\sigma_0 \cdots \sigma_l \cdots \sigma_k \cdots \sigma_t$, p_l occurs to at least the power $2\alpha_l$ contrary to (3). Next suppose $l = 0$ but $k < t$. Then $\sigma_k = a_0$, and

$$a_1 a_2 \cdots a_k a_0 = \sigma_0 \sigma_1 \cdots \sigma_{k-1} \sigma_k. \tag{8}$$

Hence from (3), numbering the p_m, $m = k+1, k+2, \cdots, t$ in any order,

$$a_{k+1} a_{k+2} a_t = \sigma_{k+1} \sigma_{k+2} \cdots \sigma_t. \tag{9}$$

But this is impossible, since

$$\begin{aligned} &\sigma_{k+1}\sigma_{k+2} \cdots \sigma_t \\ &= (1 + p_{k+1} + p_{k+1}^2 + \cdots + p_{k+1}^{\alpha_{k+1}})(1 + \cdots + p_{k+2}^{\alpha_{k+2}}) \\ &\qquad \cdots (1 + \cdots p_t^{\alpha_t}) > p_{k+1}^{\alpha_{k+1}} p_{k+2}^{\alpha_{k+2}} \cdots p_t^{\alpha_t} \\ &= a_{k+1} a_{k+2} \cdots a_t. \end{aligned} \tag{10}$$

The only remaining possibility is, then, $k = t$, $l = 0$. Thus the p's, and hence the a's, have been completely numbered as follows:

$$a_m = \sigma_{m-1}, \quad m = 1, 2, \cdots, t; \qquad a_0 = \sigma_t. \tag{11}$$

In view of (5) and Lemma 2, we must have $\alpha_0 = 1$, since otherwise

σ_0 would not be a power of a prime.

Now, evaluating the σ's, equations (11) become

$$(12)\qquad a_1 = \frac{p_0+1}{2}; \qquad a_m = \frac{p_{m-1}^{\alpha_{m-1}+1}-1}{p_{m-1}-1}, \qquad m = 2, 3, \cdots, t;$$
$$p_0 = \frac{p_t^{\alpha_t+1}-1}{p_t-1}.$$

With the definitions,

$$(13)\qquad a_m = p_m^{\alpha_m}, \qquad b_m = \frac{1}{p_m - 1}, \qquad m = 1, 2, \cdots, t,$$

equations (12) become

$$(14)\qquad a_m = b_{m-1}p_{m-1}a_{m-1} - b_{m-1} \qquad \text{for } m = 2, 3, \cdots, t,$$

$$(15)\qquad a_1 = \frac{p_0+1}{2}, \qquad p_0 = b_tp_ta_t - b_t.$$

Eliminating p_0 from (15) gives

$$(16)\qquad 2a_1 - 1 = b_tp_ta_t - b_t.$$

By repeated application of the recursion formula (14) we find that

$$(17)\qquad a_m = \left(\prod_{i=1}^{m-1} b_ip_i\right)a_1 - \sum_{j=1}^{m-1} b_j \prod_{i=j+1}^{m-1} b_ip_i, \qquad m = 2, 3, \cdots, t,$$

which is readily verified by induction. From (16) and (17) with $m=t$

$$(18)\qquad 2a_1 - 1 = \left(\prod_{i=1}^{t} b_ip_i\right)a_1 - \sum_{j=1}^{t} b_j \prod_{i=j+1}^{t} b_ip_i$$

or

$$(19)\qquad \left(\prod_{i=1}^{t} b_ip_i - 2\right)a_1 - \sum_{j=1}^{t} b_j \prod_{i=j+1}^{t} b_ip_i + 1 = 0.$$

Multiplying by $\prod_{i=1}^{t}(p_i-1)$ and using (13), (19) becomes

$$(20)\qquad \left[\prod_{i=1}^{t} p_i - 2\prod_{i=1}^{t}(p_i-1)\right]a_1 - \sum_{j=1}^{t}\left[\prod_{i=1}^{j-1}(p_i-1)\prod_{i=j+1}^{t} p_i\right] + \prod_{i=1}^{t}(p_i-1) = 0.$$

Utilizing the identity (4), (20) becomes

$$\left[\prod_{i=1}^{t} p_i - 2\prod_{i=1}^{t} (p_i - 1)\right] a_1 - \left[\prod_{i=1}^{t} p_i - 2\prod_{i=1}^{t} (p_i - 1)\right] = 0 \tag{21}$$

so that

$$(a_1 - 1)\left[\prod_{i=1}^{t} p_i - 2\prod_{i=1}^{t} (p_i - 1)\right] = 0. \tag{22}$$

Hence, either

$$p_1^{\alpha_1} = a_1 = 1 \tag{23}$$

or

$$\prod_{i=1}^{t} p_i = 2\prod_{i=1}^{t} (p_i - 1). \tag{24}$$

(23) is impossible since $p_1 \geqq 3$. (24) is also impossible, since the right member is even while the left member, being the product of odd primes, is odd. Thus the assumption that n is perfect leads to a contradiction, and the theorem is proved.

Our results may evidently be restated in the following form:

COROLLARY. *If $n = a_0 a_1 \cdots a_t$ is an odd perfect number, at least two of the divisor sums $\sigma(a_i)$ must have a common factor greater than* 1.

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