NOTE ON THE DEGREE OF CONVERGENCE OF SEQUENCES OF POLYNOMIALS

J. L. WALSH AND E. N. NILSON

The object of this note is to establish the following result:

THEOREM. *Let the power series*

$$f(z) \equiv \sum_{n=0}^{\infty} a_n z^n \tag{1}$$

have the radius of convergence ρ (>1), *and let* $p_n(z)$ *denote the polynomial of degree* n *of best approximation to* $f(z)$ *in the closed region* $|z| \leqq 1$ *in the sense of Tchebycheff. A necessary and sufficient condition for*

$$\lim_{n\to\infty} [\max |f(z) - p_n(z)|, \text{ for } |z| \leqq 1]^{1/n} = 1/\rho \tag{2}$$

is that $f(z)$ *not be of lacunary structure.*

It is well known[1] that the equation

$$\limsup_{n\to\infty} [\max |f(z) - p_n(z)|, \text{ for } |z| \leqq 1]^{1/n} = 1/\rho \tag{3}$$

is valid for every $f(z)$ defined by a power series as in (1) with radius of convergence ρ. The significance of the theorem is that the stronger relation (2) is valid except for functions of lacunary structure, as defined by Bourion.[2]

If and only if $f(z)$ is of lacunary structure, the partial sums $s_n(z) \equiv \sum_0^n a_k z^k$ are polynomials of degree n of which a suitably chosen subsequence $s_{n_k}(z)$ has the property (Bourion, loc. cit.)

$$\limsup_{n_k\to\infty} [\max |f(z) - s_{n_k}(z)|, \text{ for } |z| \leqq r]^{1/n_k} < r/\rho, \quad 0 < r < \rho. \tag{4}$$

If (4) holds for a single r, $0<r<\rho$, it holds for every such r.

If $f(z)$ is of lacunary structure, then for the extremal polynomials $p_n(z)$ of best approximation we have

$$[\max | f(z) - p_{n_k}(z)|, \text{ for } |z| \leqq 1]$$
$$\leqq [\max |f(z) - s_{n_k}(z)|, \text{ for } |z| \leqq 1],$$

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[1] Walsh, *Interpolation and approximation by rational functions in the complex domain.* Amer. Math. Soc. Colloquium Publications, vol. 20, 1935, chap. 4.

[2] *L'ultraconvergence dans les séries de Taylor,* Actualités Scientifiques et Industrielles, no. 472, 1937, pp. 9 ff.

whence from (4)

$$\limsup_{n_k\to\infty} \left[\max |f(z) - p_{n_k}(z)|, \text{ for } |z| \leqq 1\right]^{1/n_k} < 1/\rho, \tag{5}$$

and (2) is not satisfied.

Conversely, if (2) is not satisfied, then for a suitably chosen sequence n_k the first member of (5), which we denote by $1/\rho'$, is less than $1/\rho$. For values of z interior to the unit circle C we have the relations

$$f(z) - s_n(z) = \frac{1}{2\pi i}\int_C \frac{z^{n+1}f(t)dt}{t^{n+1}(t-z)},$$

$$0 = \frac{1}{2\pi i}\int_C \frac{z^{n+1}p_n(t)dt}{t^{n+1}(t-z)},$$

$$f(z) - s_n(z) = \frac{1}{2\pi i}\int_C \frac{z^{n+1}[f(t) - p_n(t)]dt}{t^{n+1}(t-z)},$$

$$\limsup_{n_k\to\infty} \left[\max |f(z) - s_{n_k}(z)|, \text{for } |z| \leqq r < 1\right]^{1/n_k} \leqq r/\rho' < r/\rho,$$

whence $f(z)$ is of lacunary structure. The theorem is established.

The generalization of equation (2), where we now consider approximation on a suitably chosen more general point set than $|z| \leqq 1$, furnishes a generalization of the concept of functions of lacunary structure.

HARVARD UNIVERSITY AND
UNITED STATES NAVAL ACADEMY