case $N^2=0$ the four sufficient conditions can be stated in the following manner. Suppose that A/N has t simple ideals. Then the Cartan invariants are the elements of a t by t matrix C with integer elements. Set $D=C-I$. Associate a graph G of $2t$ vertices $P_1, \cdots, P_t \cdot Q_1, \cdots, Q_t$ with D by joining P_i and Q_j if and only if $d_{ij} \neq 0$. The four sufficient conditions are then (1) some $d_{ij} > 1$; (2) G is not a tree; (3) some vertex of G is of order greater than 3; (4) some connected subgraph of G has more than one vertex of order 3. (Received November 21, 1946.)

23. Bernard Vinograde. *Radicals associated with equivalent semi-simple residue systems.*

This paper investigates rings wherein the radical is a homomorphic additive image of the semi-simple part and satisfies $f(xy)=xf(y)+f(x)y+f(x)f(y)$, where f is the homomorphism. $f(xy)$ affords a trioperational approach. This is an aspect of the distribution of residue systems in a semi-primary ring. (Received October 24, 1946.)

24. Daniel Zelinsky: *Nonassociative valuations.*

An ordered quasigroup G is a quasigroup, written additively, which is linearly ordered by a transitive, binary relation $>$, having the property that $x>y$ implies $x+z>y+z$ and $z+x>z+y$ for all x, y, z of G. A valuation, V, of a (nonassociative) ring R is a function on R to an ordered quasigroup with ∞ adjoined such that for all a, b of R, $V(a+b) \geqq \min [V(a), V(b)]$, $V(ab)=V(a)+V(b)$, $V(a)=\infty$ if and only if $a=0$. The principal theorem of this paper is the following: If R is an algebra of finite order over a field F, if R has a unity quantity and if $V(F)$ is an archimedean-ordered group, then $V(R)$ is an archimedean-ordered abelian group in which $V(F)$ has finite index. Examples of nonassociative ordered loops are obtained by simple loop extensions. The existence of a ring with arbitrary prescribed value loop and residue-class ring (without zero divisors) is proved. From these two facts follow examples showing that the hypothesis "$V(F)$ is archimedean-ordered" cannot be omitted in the theorem above. This is in strong contrast with the associative, noncommutative case. (See O. F. G. Schilling, *Noncommutative valuations*, Bull. Amer. Math. Soc. vol. 51 (1945) pp. 297–304.) (Received November 13, 1946.)

Analysis

25. H. W. Becker: *Generalizations of the Epstein-Fourier series.*

These series are combinations of exponential and Fourier series (Leo F. Epstein, Journal of Mathematics and Physics vol. 18 (1939) p. 60, (19)). Where $K_0=1$, $x=r\cos\theta$, $y=r\sin\theta$, and "soc" means "sine or cosine," some generalizations are: (1) $\exp[X+r \operatorname{soc}(\ )\theta\cdot(KX+Z)]=[\operatorname{soc}(Zy+e^{Xx}\sin Xy)]\cdot[\exp(Zx+e^{Xx}\cos Xy)]$, the $(KX+Z)$ except for change of sign of X being the polynomials of Steffensen (*Some recent researches in the theory of statistics and actuarial science*, Cambridge Press, 1930, p. 24); (2) $\exp[e\{1+r\operatorname{soc}(\ )\theta\cdot K^{(2)}\}]=[\operatorname{soc}\{e^{e^x\cos y}\sin(e^x\sin y)\}]\cdot[\exp\{e^{e^x\cos y}\cos(e^x\sin y)\}]$, the $K^{(2)}$ being Bell numbers (Ann. of Math. vol. 39 (1938) p. 539). Under the substitutions $r\to rX$, $X\to X^{-1}$, $Z=0$, (1) becomes (3) $\exp[X^{-1}+r\operatorname{soc}(\ )\theta\cdot T]=[\operatorname{soc}(e^x\sin y)]\cdot[\exp(e^x\cos y)]$, where $T=XKX^{-1}$ is an umbral transform of Riordan and Kaplansky (*The problem of the rooks and its applications*). It is noteworthy that the right side of (3) is free of X, endowing the left side with a kind of invariance. (Received October 1, 1946.)

26. R. H. Cameron and W. T. Martin: *Fourier-Wiener transforms of functionals belonging to L_2 over the space C.*

The authors consider the class $L_2(C)$ of real or complex-valued functionals $F(x)$ which belong to L_2 over the space C: $\int_C^w |F(x)|^2 dw < \infty$. They define a Fourier-Wiener transform and show that every functional $F(x)$ belonging to $L_2(C)$ has a transform $G(y)$ which also belongs to $L_2(C)$ and which has $F(-x)$ as its transform. They show furthermore that Plancherel's relation holds: $\int_C^w |F(x)|^2 d_w x = \int_C^w |G(y)|^2 d_w y$. In an earlier paper (*Fourier-Wiener transforms of analytic functionals*, Duke Math. J. vol. 12 (1945) pp. 489–507) the authors showed that every functional belonging to a restricted class E_m has a Fourier-Wiener transform which belongs to E_m and which has the properties mentioned above. (The definition given there is modified slightly in the present paper to free Plancherel's formula from a factor $2^{1/2}$.) The functionals of class E_m are shown to be dense in $L_2(C)$; this property enables the authors to obtain the desired result. (Received November 6, 1946.)

27. K. L. Chung and Paul Erdös: *On the lower limit of sums of independent random variables.*

The following theorems form a counterpart of the law of iterated logarithms. Let $X_1, \cdots, X_n, \cdots$ be independent random variables having the same distribution function $F(x)$. Suppose that the absolutely continuous part of $F(x)$ does not vanish identically and that its first moment is zero, the second is one, and the absolute fifth is finite. Let $\psi(n)$ be nondecreasing and tend to infinity. Then (1) $\Pr(\lim \inf_{n\to\infty} n^{1/2}\psi(n)S_n = 0)$ is one or zero according as (2) $\sum n^{-1}\psi(n)^{-1}$ is divergent or convergent. If each X_n has the same Bernoullian distribution: $X_n = q$ with probability p and $X_n = -p$ with probability q where $p>0$ and $p+q=1$, then if (2) is divergent, (1) is equal to one. On the other hand, if p is a quadratic irrational, and (2) is convergent, (1) is equal to zero. The last statement holds in fact for almost all real p. The case when p is rational is known to be degenerate. (Received October 31, 1946.)

28. Paul Civin: *Mean values of periodic functions.*

Let $f(x)$ be a complex periodic function of period 2π for which the Fourier coefficients of order less than m vanish. Inequalities are obtained between the integral means of $f(x)$ and transforms of $f(x)$ by means of factor sequences. As a specialization, inequalities are obtained between the means of $f(x)$ and its αth integral. (Received October 28, 1946.)

29. Evelyn Frank: *On continued fraction expansions of analytic functions.*

(i) A continued fraction expansion $\gamma_0 + \sigma_0(z+\bar{\xi}_0)(1-|\gamma_0|^2)/\sigma_0\bar{\gamma}_0(z+\bar{\xi}_0) - (z-\xi_0)/\gamma_1 + \sigma_1(z+\bar{\xi}_1)(1-|\gamma_1|^2)/\sigma_1\bar{\gamma}_1(z+\bar{\xi}_1) - (z-\xi_1)/\gamma_2 + \cdots$, $\gamma_k = f_k(-\bar{\xi}_k)$, $f_{k+1} = \sigma_k(z-\xi_k)(\gamma_k - f_k)/(z+\bar{\xi}_k)(1-\bar{\gamma}_k f_k)$, $k=0, 1, \cdots$, for functions $f(z)$ analytic in $R(z)>0$, $|f(z)| \leqq 1$, is found directly from Schur's theory by mapping $|w|<1$ on $R(z)>0$. Here $\xi_k = \xi$, $R(\xi)<0$, $\sigma_k = 1$, $|\gamma_k| \leqq 1$. This expansion is proved finite by use of the reduction theorem (cf. Frank, *The real parts of the zeros of a complex polynomial*, abstract 52-7-232) if and only if $f(z) = \epsilon P(z)/P^*(z)$, $|\epsilon| = 1$, $P^*(z) = \bar{P}(-z)$, $P^*(z)$ a Hurwitz polynomial, $P(-\bar{\xi}) = 1$. If $P^*(z)$ is any polynomial, then in the finite continued fraction $|\sigma_k| = |\bar{P}_k^*(-\xi)/P_k^*(-\bar{\xi}_k)| = 1$, the ξ_k not necessarily equal. (ii) The continued fraction $[z+(\bar{\xi}-\xi)/2]/(\bar{\xi}-\xi)/2 + (z+\bar{\xi})(z-\xi)(\delta_0-\gamma_0)/[\delta_0(z-\xi) - \delta_1(z+\bar{\xi})] + (z+\bar{\xi})(z-\xi)(\delta_1

$+\gamma_0)(\delta_1-\gamma_1)/[\delta_1(z-\xi)-\delta_2(z+\bar{\xi})]+(z+\bar{\xi})(z-\xi)(\delta_2+\gamma_1)(\delta_2-\gamma_2)/[\delta_2(z-\xi)-\delta_3(z+\bar{\xi})]+\cdots$, $\delta_{k+1}=(\delta_k-\gamma_k)/(1-\delta_k\bar{\gamma}_k)$, $k=0, 1, \cdots$, converges uniformly over every bounded closed domain in $R(z)>0$ to a function, analytic in $R(z)>0$, $F(z)=[\delta_0(z-\xi)+(z+\bar{\xi})f(z)]/[\delta_0(z-\xi)-(z+\bar{\xi})f(z)]$, $|\delta_0|\leqq 1$, $R(\xi)<0$. (iii) Functions $F(z)$ and $[\lambda-f(z)]/[\lambda+f(z)]$, where $f(z)=\epsilon P(z)/P^*(z)$, have partial fraction expansions $Q(z)+c-\sum_{v=1}^{n-k}r_v(z+\eta_v)/(z-\eta_v)$, $Q(z)$ zero or a polynomial of degree k, the η_v zeros of the denominator of the function represented, and under certain conditions the c and r_v pure imaginary. (Received October 17, 1946.)

30. Evelyn Frank: *On methods for the location of the zeros of complex polynomials.*

(I) The zeros of $P(z)\equiv A_0z^n+A_1z^{n-1}+\cdots+A_n$, A_j complex, are located as follows: If $|A_0|>|A_n|$, form $\bar{A}_0P(z)-A_nP^*(z)\equiv zP_1(z)$, $P^*(z)\equiv z^n\bar{P}(1/z)$, then $P(z)$ has one more zero in $|z|=1$ than $P_1(z)$; if $|A_0|<|A_n|$, $P_1(z)\equiv\bar{A}_nP(z)-A_0P^*(z)$ has as many zeros as $P(z)$ in $|z|=1$. Repeating this process on $P_1(z)$, $P_2(z)$, $\cdots$, after at most $(n-1)$ steps one finds the number of zeros in $|z|=1$ (Cohn, Math. Zeit. vol. 14). Form $P(rz)$ and vary r successively until, by this process, $P_{n-1}(rz)$ has a zero z' of modulus 1 (or as close to 1 as desired), then rz' is a zero of $P(z)$ of modulus r (or as close to r as desired). The exceptional cases $|A_v^{(k)}|=|A_{n-v}^{(k)}|$, $v=0, 1, \cdots$, in $P_{n-k}(z)$ are considered. The zeros of $P(z)$ are different and of modulus r if $A_n\bar{A}_v=\bar{A}_0A_{n-v}r^{2v}$, $v=1, 2, \cdots, n$, and the zeros of $P'(rz)$ lie in $|z|=1$. (II) An alternate method for finding the zeros of $P(z)$ is obtained by use of the quadratic forms of Hermite. (III) Schur's determinant criterion (J. Reine Angew. Math. vol. 148) for all zeros of $P(z)$ to lie in $|z|=1$ is compared with that of the author (Bull. Amer. Math. Soc. vol. 52) for all zeros to lie in $R(z)>0$. (Received November 20, 1946.)

31. A. M. Gleason: *A theorem on Banach spaces.* Preliminary report.

The sequence $\{\phi_n\}$ of points in a Banach space B is called a basis if every element $x\in B$ has a unique convergent representation $x=\sum a_n(x)\phi_n$. A basis is called an absolute basis if the convergence of $\sum a_n\phi_n$ implies the convergence of $\sum a_{n_i}\phi_{n_i}$ for any sequence $\{n_i\}$. Under this definition the ordinary basis of l_p $(1\leqq p<\infty)$ is absolute, but Schauder's basis for the set of continuous functions is not. Theorem: If B has an absolute basis and B^* (the conjugate of B) is separable, then B^* has an absolute basis. It is well known that the functionals $a_n(x)$ form a basis of their closed linear manifold; if there is a functional f not in this clm, one can construct an uncountable set of points in B^* each pair of which are separated by more than $h>0$, so that B^* is not separable. This construction is obtained by considering functionals $g(x)=\sum b_{n_i}a_{n_i}(x)$, where $f(x)=\sum b_na_n(x)$. An immediate corollary is the following: If B has an absolute basis and B^{**} is separable, then B is reflexive. (Received November 22, 1946.)

32. O. E. Glenn: *A differential equation that plays a part in a mathematical theory of biological variation.*

A sequence of plane curves k_1, k_2, $\cdots$ is considered, in the form of irregular (statistical) truncated spirals of amplitude less than four right angles. Draw k_1 in the (y, z) cartesian plane and k_2 in the (x, y) plane. The cylinders erected upon these curves as bases with elements $\|x$ and z, respectively, intersect in a twisted space-curve c_1. Move k_2 to the plane (y, z) and draw k_3 in (x, y) and repeat the construction, giving a space-curve c_2, and continue the process. The curves of the *community* (c_j) are permuted among themselves by a known transformation S in polar coordinates (Annali della R. Scuola Normale Superior di Pisa (2) vol. 2 (1933) p. 300), and a function $I(r, \theta, \phi, d\theta, d\phi)$ left invariant by S satisfies the equation (1) $(up\, \partial/\partial r + vq\partial/\partial\theta + up'(dr)\partial/\partial(dr) + vq'(d\theta)\partial/\partial(d\theta))I = 0$, $p = ar^{l-1} + br^{l-2} + \cdots + k$, $q = \alpha\theta^{l-1} + \beta\theta^{l-2} + \cdots + \kappa$, $u, v = 0$. Solution for I and reduction of $I = 0$ gives (2) $-(dr)/p + \lambda(Q, \phi)(d\theta)/q + \mu(Q, \phi)d\phi = 0$, λ, μ arbitrary, $Q = v\int(dr)/p - u\int(d\theta)/q$. Equation (2) is formally invariant under S and its integral surfaces are related, by theorems, to the community (c_j). (Received November 8, 1946.)

33. Casper Goffman: *Fubini interval functions.*

A function $F(R)$ of rectangles and a function $f(I)$ of line segments, both properly oriented, satisfy a condition of Fubini if $F(R) = \int f(I)dy$. Assuming that $F(R)$ and $f(I)$ satisfy a condition of Fubini, and if E, not necessarily measurable, is the set of points for which both $f'(x, y)$, the bilateral derivative of $f(I)$, and $F_S'(x, y)$, the strong derivative of $F(R)$, exist, then $f'(x, y) = F_S'(x, y)$ almost everywhere on E. A similar result for the unilateral derivative is proved for E measurable, but the theorem then holds for the approximate unilateral derivative. It is shown by example that these results do not hold for the ordinary derivative, $F'(x, y)$, of $F(R)$. Conditions are established under which the existence of $f'(x, y)$ implies the existence almost everywhere of $F'(x, y)$ and $F_S'(x, y)$. The existence of $F_S'(x, y)$ implies the existence of the approximate unilateral derivative and its equality almost everywhere with $F_S'(x, y)$. A similar result is obtained for the derivative of $f(I)$ only by assuming continuity. (Received November 21, 1946.)

34. M. R. Hestenes. *An indirect sufficiency proof for the problem of Bolza in nonparametric form.*

Until recently, sufficiency theorems in the calculus of variations have been established by direct methods. Indirect proofs have now been devised by McShane (Trans. Amer. Math. Soc. vol. 52 (1942) pp. 344–379), by Myers (Duke Math. J. vol. 10 (1943) pp. 73–99) and by the author (Trans. Amer. Math. Soc. vol. 60 (1946) pp. 93–118). Sufficient conditions for a strong relative minimum for the nonparametric case can be obtained from the results given in these papers by transforming our problem to a parametric problem. However the methods used are not applicable directly to the nonparametric case without modification. It is the purpose of the present paper to show what modifications are necessary in order to establish the corresponding sufficiency theorem for the nonparametric problem without transformation to a parametric one. (Received October 29, 1946.)

35. Dunham Jackson: *The boundedness of orthonormal polynomials on certain curves of the fourth degree.*

For the study of the boundedness of systems of orthonormal polynomials on algebraic curves, a property intimately associated with the convergence of the corresponding developments in series, various elementary devices and expedients have been found useful under one set of circumstances or another. The general problem is of such complexity that there is no reason to expect that it can be dealt with by any combination of such special devices. On the other hand, even if attention is limited to these particular procedures, any attempt at an exhaustive account of their possibilities in permutation and combination would be unmanageable. It seems worthwhile to continue at least a little further with an analysis of illustrative examples. In this spirit the writer discusses certain rational curves of the fourth degree, and also the nonrational curve $x^4+y^4=1$, previously studied from a different point of view (Bull. Amer. Math. Soc. vol. 43 (1937) pp. 388–393). While further examples could be multiplied without limit, it is the writer's impression that, in a certain qualitative sense, an enumeration of the simplest and most striking illustrations of the methods referred to is hereby concluded. (Received October 30, 1946.)

36. Herman Kober: *Approximation of non-bounded functions by integral functions of finite order.*

The introduction deals with a theorem by Carleman; it also mentions a new result concerning approximation to bounded functions. The non-bounded functions are required to be uniformly continuous of order n, that is, $|\sum(-1)^n C_{nj} f(x+jh)| \leqq \epsilon$ $(j=0, 1, \cdots, n)$ for $-\infty < x < \infty$, $|h| \leqq \delta = \delta(\epsilon)$, or at least to be "uniformly bounded of order n." After discussing basic properties of these classes of functions, the following results are proved: If $f(z)$ is an entire function of exponential type, and if $f(x)$ is uniformly bounded of order n, then $f^{(n)}(x)$ is bounded in $(-\infty, \infty)$. A function $f(x)$, uniformly bounded of order n, is approximated by entire functions of exponential type uniformly in $-\infty < x < \infty$ if, and only if, it is uniformly continuous of order n. If $(x+i)^m f(x)$ $(n=1$, or $2, \cdots)$ is uniformly continuous of some order, then there are entire functions $g_\alpha(z)$ of exponential type α such that, uniformly in $-\infty < x < \infty$, $(1+|x|^m)\ |f(x)-g_\alpha(x)| \to 0$ as $\alpha \to \infty$. The results are applied to $f(x)=x^r$ $(0 < r < \infty)$, also for approximation by entire functions of order 1/2. They are finally extended to the uniform approximation of non-bounded functions, analytic in the half-plane and satisfying certain conditions, by entire functions of exponential type. (Received October 16, 1946.)

37. C. E. Langenhop: *Analytic functions of matrices.*

A number of definitions of analytic functions of matrices have been given. In this paper square matrices are treated as numbers and the definitions of ordinary calculus with a few necessary modifications are applied to them, an approach which is essentially different from those reviewed by MacDuffee (*The theory of matrices*, Ergebnisse der Mathematik and ihrer Grenzegebiete, vol. 2, no. 5, pp. 99–101). Definitions of the terms function, derivative, and analyticity as applied to matrices are given. Second order matrices are extensively treated and necessary and sufficient conditions for one matrix to be an analytic function of another, including differential equations analogous to the Cauchy-Riemann equations, are derived. These conditions are utilized to solve the matrix differential equation $D_X f(X) = f(X)$, and the elements of the matrix function $f(X)$ are found in closed form. (Received October 17, 1946.)

38. J. D. Mancill: *Unilateral variations with variable end points.*

The author studies the properties of an arc E_{12} in order that it minimize an integral $I \equiv \int_{t_2}^{t_1} F(x, y, x', y')dt$ in the class of all admissible curves joining a given curve D through the point 1 and a given curve C through the point 2. The entire arc E_{12} or any part of it may lie along the boundary of the region R of admissible curves. A simple derivation of the transversality condition is given which requires weaker hypotheses than the usual proof by means of the first variation and is independent of the Euler equations. The transversality condition involves an inequality at an end point which lies on the boundary of R. A set of sufficient conditions is given for each of the several cases, which permit the curves D and C to be tangent to a non-extremal arc E_{12}. The treatment given for the case when E_{12} is an extremal and the curves D and C are transversal to E_{12} applies to free variations and is much simpler than that given by Bliss (*Jacobi's criterion when both end points are variable*, Math. Ann. vol. 58 (1904) pp. 70–80). The entire results of the paper are applicable to the problem in nonparametric form. (Received October 17, 1946.)

39. A. D. Michal: *An existence and uniqueness theorem for a non-linear differential equation in Banach spaces.*

Let B_1 and B_2 be Banach spaces, and let $T(y_1, x, y_2)$ be a trilinear function on $B_1B_2B_1$ to B_1. A global existence and uniqueness theorem is given for the differential system in Fréchet differentials $\delta y(x) = T(y(x), \delta x, y(x))$, $y(x_0) = y_0$. Under some natural restrictions on T, the solution $y(x)$ is shown to have a generalized Taylor's series expansion in successive Fréchet differentials valid for *all* $x \in B_2$. This theory is applied to "ordinary" linear differential equations in Banach spaces, to the matrizant functional, and to related topics. The situation is also studied in complex Banach spaces. (Received October 24, 1946.)

40. A. D. Michal: *Global groups of motions in some infinitely dimensional Riemannian spaces.*

The author shows, by furnishing examples, that there exist global continuous groups of motions in some infinitely dimensional Riemannian spaces with a variable (infinitely dimensional) Riemannian curvature. (Received October 24, 1946.)

41. A. D. Michal: *The solutions of systems of linear differential equations as entire analytic functionals of the coefficient functions.*

The results of a previous paper (Bull. Amer. Math. Soc. Abstract 53-1-39) are used to obtain theorems on the subject matter of the title. These investigations have important applications to the approximate solution of systems of linear differential equations with variable coefficients and stem from the generalized Taylor's series expansions for the solutions. (Received October 24, 1946.)

42. A. D. Michal, R. C. James, and Max Wyman: *Topological abelian groups with ordered norms.*

A study is made of the minimal normed linear spaces that contain such groups. A theory of differentials of functions is given. The theory is related to Fréchet differentials in normed linear spaces and to M_1 differentials in topological abelian groups. (Received October 24, 1946.)

43. W. D. Munro: *Orthogonal trigonometric sums with auxiliary conditions.*

The author considers the construction of a system of trigonometric sums, orthonormal over a period interval and subject to certain auxiliary conditions of the form $U_i(f) = 0$, the $U_i(f)$ being linear, homogeneous functionals. Conditions for convergence of a series of such sums at an arbitrary point are obtained, and an analogue of a theorem of Korous on the boundedness of orthogonal polynomials is proved. (Received October 21, 1946.)

44. M. O. Reade: *On areolar monogenic functions.*

In this paper, results similar to those derived by Haskell (Bull. Amer. Math. Soc. vol. 52 (1946) pp. 332–337) are obtained. Some of the results are extensions of Haskell's results, and some pertain to areolar monogenic polynomials of certain form. This paper will appear in the Bulletin. (Received November 26, 1946.)

45. P. C. Rosenbloom: *Studies on the heat equation.*

New results on the uniqueness and representation of solutions of the heat equation are obtained, extending and sharpening results of Tykhonov, Widder, and Pollard. It is shown that methods which have been used in the theory of harmonic functions can be applied to the study of the heat equation. Thus, theorems analogous to the Fatou theorem and the Phragmèn-Lindelöf theorem are obtained. (Received November 21, 1946.)

46. Robert Schatten: *An evaluation of the bound for certain functionals.*

The present notation is that of Bull. Math. Soc. Abstract 51-9-162. The bound on $\mathfrak{B}_1 \otimes_\gamma \mathfrak{B}_2$ of an invariant under equivalence functional $\mathfrak{F}(\sum_{i=1}^n f_i \otimes g_i)$ of expressions which satisfy the condition $|\mathfrak{F}(\sum_{i=1}^n f_i \otimes g_i \cdot + \cdot \sum_{j=1}^m h_j \otimes k_j)| \leqq |\mathfrak{F}(\sum_{i=1}^n f_i \otimes g_i)| + |\mathfrak{F}(\sum_{j=1}^m h_j \otimes k_j)|$ can be expressed in terms of expressions of $f \otimes g$ of rank 1 as $\sup |\mathfrak{F}(f \otimes g)| / \|f\| \, \|g\|$ for $f \in \mathfrak{B}_1$, $g \in \mathfrak{B}_2$; $\|f\| = \|g\| = 1$. In particular if the S_i and T_i denote linear transformations on $\mathfrak{B}_1$ and $\mathfrak{B}_2$ respectively, then for any crossnorm α, $\sum_{i=1}^n S_i \otimes T_i$ represents a linear transformation from $\mathfrak{B}_1 \otimes_\gamma \mathfrak{B}_2$ into $\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2$, with a bound equal to $\sup \alpha(\sum_{i=1}^n S_i f \otimes T_i g) / \|f\| \, \|g\|$. (Received November 20, 1946.)

47. I. J. Schoenberg: *On totally positive functions, Laplace integrals and entire functions of the Laguerre-Pólya-Schur type.*

A real-valued function $\Lambda(x)$, defined for all real x, is said to be *totally positive* if it satisfies the following three conditions: $(\alpha)\,\Lambda(x)$ is measurable. (β) If $x_1 < x_2 < \cdots < x_n$ and $t_1 < t_2 < \cdots < t_n$, then the inequality $\det \|\Lambda(x_i - t_j)\| \geqq 0$ should hold. This condition should be satisfied for $n = 1, 2, \cdots$. $(\gamma)\,\Lambda(x)$ should be different from zero for at least two values of x. A trivial example of such functions is $\Lambda(x) = \exp(ax+b)$. G. Pólya (*Algebraische Untersuchungen über ganze Funktionen vom Geschlechte Null und Eins*, J. Reine Angew Math. vol. 145 (1915) pp. 224–249) has raised the question as to the possibility of representing by a bilateral Laplace integral the reciprocal $1/\Psi(z)$ of an entire function $\Psi(z)$ which is the uniform limit of a sequence of real polynomials with only real roots. It is shown here $\epsilon/\Psi(z)$, where $\epsilon = 1$ or -1 and $\Psi(z) \not\equiv \exp(az+b)$, allows in every one of its vertical strips of regularity a representation $\int_{-\infty}^{\infty} \exp(xz)\Lambda(x)dx$, where $\Lambda(x)$ is totally positive. Conversely, a Laplace integral of this type, where $\Lambda(x) \not\equiv \exp(ax+b)$, always converges in a vertical strip and represents there a reciprocal $1/\Psi(z)$. (Received November 18, 1946.)

48. M. H. Stone: *Pseudo-norms and partial orderings in abelian groups.*

In this paper it is shown that if an abelian group X has a pseudo-norm then it can be partially ordered and provided with a new pseudo-norm which is equivalent to the old and naturally related to the ordering introduced. The case where X is a Banach space and the case where the pseudo-norm for X is a norm are both discussed, the latter being fully characterized. (Received October 15, 1946.)

49. W. M. Stone: *On Samuelson's generalized Laplace transformation.*

P. A. Samuelson (Bull. Amer. Math. Soc. Abstract 52-3-86) pointed out that a generalized Laplace transformation, $L(s:f(t))_E = \sum_0^\infty f(n)s^{-n-1}$, yields operational methods for solution of difference equations exactly like those for differential equations. Further applications are developed in this paper. Relationships between sums and integrals which define convolution in both the generalized and ordinary sense are brought out by use of a table of function pairs such that the ordinary transform of one is equal to the generalized transform of the other. The summation of certain Taylor and Fourier series is reduced to an integration and a real inversion formula corresponding to that of Post (Trans. Amer. Math. Soc. vol. 32 (1930) pp. 723–781) is developed from the analogy with a Taylor series expansion. (Received October 21, 1946.)

50. Walter Strodt (National Research Fellow): *On the principal solution of a difference equation.* Preliminary report.

Let (1) $f[x, y(x+\omega_1), \cdots, y(x+\omega_n)] = \phi(x)$ be a difference equation, with $\phi(x)$ a given function of the complex variable x, analytic throughout a region $\mathcal{B}$ containing the position axis, with f a given polynomial in y having coefficients analytic throughout $\mathcal{B}$ and having no term free of y, and with $\omega_1, \cdots, \omega_n$ given complex numbers. Let $q_1, \cdots, q_n$ be complex functions of any positive number b such that $b(1-q_k)$ tends to ω_k as b becomes infinite. Let t be a non-negative integer. A *special solution* $y(x, b)$ of (2) $f[x, y(q_1(x-b)+b), \cdots, y(q_n(x-b)+b)] = (1-x/b)^t\phi(x)$ is any series of *real* powers of $x-b$ satisfying (2) in a circle with center b, cut along the right-most radius. A *principal solution* of (1) is any $y(x)$ which in some suitable subregion of $\mathcal{B}$ is the uniform limit of an analytic continuation of $y(x, b)$, as b becomes infinite on some sequence of positive numbers. This principal solution is shown to be a broad generalization of the principal solution as defined and studied by Nörlund (*Differenzenrechnung*, 1924). It is conjectured that the principal solution becomes the totality of analytic solutions if the word "real" in the definition of special solution is replaced by "complex." (Received October 17, 1946.)

51. Otto Szász: *On closed sets of rational functions.*

Theorem 1. Let $\{z_n\}$ be a sequence of points in the half-plane $Rz>0$; a necessary and sufficient condition that either of the sequences $\{(x^2+z_n^2)^{-1}\}$, $\{x(x^2+z_n^2)^{-1}\}$ be closed in $L_2(0, \infty)$ is that $\sum Rz_n/(1+|z_n|^2)=\infty$. Theorem 2. Let $\{\zeta_n\}$ be a sequence of points in the unit circle $|z|<1$; the sequence of functions $\{(1-\zeta_n z)^{-1}\}$ is closed in H_2 if and only if $\sum(1-|z_n|)=\infty$. The method used for the proof of these theorems is elementary and essentially the same as used partly by Müntz and in a simpler and more general form by the author to solve the analogous problem for the sequence $\{x^{\lambda_n}\}$ in $L_2(0, 1)$. Closure and completeness in other spaces and for other sequences of functions are also considered. (Received November 9, 1946.)

52. Gabor Szegö: *On an inequality due to P. Turan concerning Legendre polynomials.*

P. Turan proved recently the following theorem. Let $P_n(x)$ be the nth Legendre polynomial. Then the inequality $P_{n-1}(x)P_{n+1}(x)-(P_n(x))^2\leqq 0$ holds in the interval $-1\leqq x\leqq 1$, with equality only for $x=\pm 1$. For this theorem three proofs are given based on different principles. (Received October 24, 1946.)

53. H. P. Thielman: *On continuous and discontinuous solutions of generalized Cauchy functional equations.*

One of the functional equations considered is reduced to the form $f(x+y+nxy) = f(x)+f(y)$, $n>0$, $x>-1/n$, $y>-1/n$. It is shown that every continuous solution of this equation is of the form $f(x)=k\log(1+nx)$, where k is any real number. On the basis of Zermelo's postulate that the real continuum can be well ordered, discontinuous solutions of this equation are constructed. The method used is similar to the one used by Hamel (Math. Ann. vol. 60 (1905) pp. 459–462). Some of the results obtained by Blumberg (Trans. Amer. Math. Soc. vol. 20 (1919) pp. 40–44), Sierpinski (Fund. Math. vol. 1 (1920) pp. 116–122), and Ostrowski (Comment. Math. Helv. vol. 1 (1929) pp. 157–159) for convex functions are shown to apply to the solutions of the given functional equation. In particular, every solution which is measurable is continuous. Other functional equations considered can be reduced to the following forms: $g(x+y+nxy)=g(x)\cdot g(y)$, $h(x)+h(y)+nh(x)h(y)=h(x+y)$, and $k(x)+k(y)+nk(x)k(y) = k(xy)$. (Received October 17, 1946.)

54. W. J. Thron. *Periodic iterated fractions.*

Let $w=T_n(z)$ be an arbitrary linear fractional transformation of the z-plane into the w-plane. Let $F_n(z)\equiv T_1T_2\cdots T_n(z)$. Then $\{F_n(z)\}$ is called an "infinite iterated fraction." In this paper necessary and sufficient conditions for the convergence of periodic infinite iterated fractions ($T_{rk+m}=T_m$ for all $r\geqq 0$; $m=1, 2, \cdots, k$) are given. Further a theorem analogous to the Galois-Pringsheim theorem for periodic continued fractions is proved for iterated fractions. (Received October 25, 1946.)

55. Leonard Tornheim: *Harmonic series in two variables.*

The value of $\sum_{p,q=1}^{\infty}1/p^mq^r(p+q)^s$ in terms of $c_n=\sum_{p=1}^{\infty}1/p^n$ is found for $t=m+r+s$ odd and for t even under any of the following conditions: m, r, or s is 1; $m=r=s$; $s=0$; $r=0$ and $m=s$; or $s=r+1=m+2$. The following result is used: if $f(p)$ is a monotone function decreasing to zero, then $\sum[2f(q)/p(p+q)-f(p+q)/pq]=2f(1)$, where the sum is for pairs p, q which are positive and relatively prime. (Received November 21, 1946.)

56. W. J. Trjitzinsky: *Singular integral equations with complex-valued kernels.*

This work, to appear in the Annali di Matematica, is an extensive investigation of various types of integral equations with singular kernels, which are limits of complex-valued regular kernels, having possibly complex-valued characteristic values and real characteristic functions; use is made of a suitable spectral theory. (Received November 20, 1946.)

57. C. A. Truesdell: *On a class of differential-difference equations.*

The equation $\partial f(x, \alpha)/\partial x = G(f(x, \alpha+n), f(x, \alpha+n-1), \cdots, f(x, \alpha), x, \alpha)$ is studied. Existence and uniqueness theorems for the boundary condition $f(x_0, \alpha) = \phi(\alpha)$ are given, where $\phi(\alpha)$ is defined either at the points $\alpha = \alpha_0 + i$, $i = 0, 1, 2, \cdots$, or in a right half complex plane. It is mentioned that the earlier proof of Bruwier (Liége Memoires (3) vol. 17 (1932) pp. 1–48) is inadequate because it treats only the case $\phi = \text{const}$. Proofs are by successive approximation, generalizing the existence proof for an ordinary differential equation of first order. The existence of periodic solutions is discussed. If the function G is periodic of period 1 in α, it is shown that a solution satisfying the condition $f(x_0, \alpha) = \phi(\alpha)$ is periodic of period 1 in α if and only if $\phi(\alpha)$ is periodic of period 1. Examples show that if G is not periodic the periodicity of ϕ is not sufficient to insure the periodicity of the corresponding solution f, and if G is periodic, but ϕ is not, the corresponding solution is not necessarily periodic, so the above theorem is the strongest possible. It is shown, further, that a necessary and sufficient condition for the existence of a single periodic solution is that $G(y, y, \cdots, y, x, \alpha+1) = G(y, y, \cdots, y, x, \alpha)$. (Received November 20, 1946.)

58. Alexander Weinstein: *On the generalized Stokes-Beltrami equations.*

Let $\phi(x, y)$ and $\psi(x, y)$ be a pair of solutions of the equations (*) $y\Delta\phi + p\phi_y = 0$, $y\Delta\psi - p\psi_y = 0$, where p is a non-negative real number and ϕ and ψ can be interpreted as an axially symmetric potential and the corresponding stream function in a "space" of $n = 2 + p$ dimensions. The author establishes the obvious identity $y^{k-1}(y\Delta f + kf_y) = y\Delta g + (2-k)g_y$ where $g(x, y) = y^{k-1}f(x, y)$. This identity shows, for $k \geqq 0$, the connection between the solutions of (*) in spaces of different dimensions. This result combined with the divergence theorem for the potential ϕ yields a new approach to the theory of a class of discontinuous integrals involving Bessel functions (see A. Weinstein, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 240, 431, 823). (Received November 18, 1946.)

59. J. E. Wilkins: *The growth of the solutions of a certain linear integro-differential equation.*

The equation to be solved is of the form $d\phi/dt = -\lambda \int K(u)\phi(t-u)du$, in which the integration is extended over the interval $(0, a)$. Here a and λ are positive constants. Preliminary results on the growth of solutions $\phi(t)$ which are of finite exponential type are obtained when the kernel $K(u)$ is positive, as it usually is in the physical applications. More complete results are then obtained when $K(u)$ is subjected to hypotheses of monotonicity and convexity. (Received November 18, 1946.)

60. František Wolf: *On a decomposition of functions.*

A function $f(x)$ will be called of finite class $\mathcal{C}$, if there exist m and n such that $\int_0^x |x-y|^m f(y)dy = O(|x|^n)$. Such a function can be expressed by means of a generalized integral of the Bochner type which is (C, k) summable to $f(x)$ for large enough k's. In this sense it has a meaning to speak of functions of $\mathcal{C}$ with a non-negative spectrum. Any function of $\mathcal{C}$ can be expressed as the sum of two functions f_1, f_2 of $\mathcal{C}$ of which one has a non-negative and the other a nonpositive spectrum. The decomposition is unique up to a polynomial. f_1 is the boundary function of a function of $\mathcal{C}$ analytic in the upper half-plane, f_2 has the same property w.r.t. the lower half-plane. From here we obtain a result by T. Carleman (*L'integrale de Fourier*, Uppsala, 1944, p. 42). (Received October 19, 1946.)

61. J. W. T. Youngs: *Remarks on the isoperimetric inequality for closed Fréchet surfaces.*

The author treats the particular isoperimetric inequality $36\pi V^2 \leqq A^3$, where A is the Lebesgue area of a closed Fréchet surface S, and V is the volume enclosed by S. These terms are defined in a forthcoming paper by Radó, who establishes the inequality for every S. In the present paper, the purpose is twofold: first, to offer a variation of the definition of enclosed volume which generally increases the left side of the inequality; second, to investigate the consequences of equality, under both definitions of enclosed volume. The basic concept employed is that of the cyclic decomposition of a closed Fréchet surface S. It is shown, for example, that if $36\pi V^2 = A^3$ and $0 < A < \infty$, the cyclic decomposition of S reduces to precisely one closed surface. (Received November 20, 1946.)

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62. N. R. Amundson: *Unsymmetrically loaded orthotropic thin plates on elastic foundations.*

The author considers orthotropic thin plates of infinite extent on elastic foundations of two different kinds. In the first case the foundation exerts a reactive pressure proportional to the deflection. The solution of the plate equation for this case, $D(u) = au_{xxxx} + 2bu_{xxyy} + cu_{yyyy} = q(x, y) - ku(x, y)$ with suitable conditions at infinity, is $u(x, y) = (4\pi^2)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\alpha, \beta)d\alpha d\beta \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} q(s, t)\exp(-i[\alpha(x-s)+\beta(y-t)])dsdt$ where $q(x, y)$ is the loading function and $F(\alpha, \beta) = (a\alpha^4 + 2b\alpha^2\beta^2 + c\beta^4 + k)^{-1}$. In the second case the foundation is the homogeneous istropic semi-infinite medium of classical elasticity theory (Timoshenko, *Theory of elasticity*, p. 332). The solution of the plate equation for this case, $D(u) = q(x, y) - p_s(x, y)$, where $p_s(x, y)$ is the reactive pressure of the foundation, is obtained from an equivalent integral equation, $u = k\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[q(s, t) - D(u)]rdsdt$, and is $u(x, y) = (4\pi^2)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\alpha, \beta)d\alpha d\beta \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} q(s, t)\exp(-i[\alpha(x-s) + \beta(y-t)])dsdt$ where $r = [(x-s)^2 + (y-t)^2]^{-1/2}$ and $G(\alpha, \beta) = [(\alpha^2+\beta^2)^{1/2}(2\pi k)^{-1} + a\alpha^4 + 2b\alpha^2\beta^2 + c\beta^4]^{-1}$. These solutions are obtained by the use of the double Fourier transform. The solution of the first problem can also be obtained from the finite plate as a limiting case. Various special cases are considered. This work is connected with that of Holl, Hogg, Woinowski-Krieger, Happel, Lewe, et al. (Received October 25, 1946.)

63. J. W. Beach: *Flow of slow viscous fluid between rotating cylinders.*

A solution is obtained for the biharmonic equation in the region between eccentric cylinders when the stream function and its normal derivative are given on both cylinders. The solution is set up in terms of two functions of a complex variable and a transformation of co-ordinates is made so that the bounding cylinders become concentric. The form of the two functions is determined with the single-valued parts given as infinite series. Unknown coefficients are determined and the solution obtained. The limit of this solution as the cylinders become concentric is obtained and agrees with the known solution for concentric cylinders. (Received October 23, 1946.)