A GENERALIZATION OF A THEOREM OF LEROY AND LINDELÖF

V. F. COWLING

1. Introduction. Consider a Taylor series $f(z)=\sum_{n=0}^{\infty}a_n z^n$ with radius of convergence unity. Let the coefficients a_n be the values taken on by a regular function $a(z)$ for $z=0, 1, \cdots$.

The object of this paper is to study the Taylor series under the assumption that $a(z)$ is regular in certain domains.[1] The results obtained are of the nature of domains in which the function defined by $\sum_{n=0}^{\infty}a_n z^n$ is regular and of domains which contain the singularities of the function defined by the series. In terms of $a(z)$ fairly general sufficient conditions are given such that the circle of convergence is not a cut for the function.

The results may be regarded as a generalization of a theorem due to LeRoy and Lindelöf.[2]

THEOREM (LEROY AND LINDELÖF). *Suppose* (a) $a(x+iy)$ *is regular in the semiplane* $x\geqq\alpha$, (b) *there is a* $\theta<\pi$ *such that for every arbitrary small positive* ϵ *and for sufficiently large* ρ

$$|a(\alpha+\rho\exp(i\psi))| < \exp[\rho(\theta+\epsilon)], \qquad -\pi/2 \leqq \psi \leqq \pi/2.$$

Then

$$f(z) = \sum_{n=0}^{\infty} a(n)z^n, \qquad z = r\exp(i\phi)$$

is regular in the angle

$$\theta < \phi < 2\pi - \theta.$$

The generalization of this theorem that we prove consists, under suitable restrictions, in replacing the semiplane $x\geqq\alpha$ by an angular opening including the axis of positive reals in its interior.

The singularities of the function $f(z)$ studied in this paper are those of a "principal branch" obtained by immediate continuation of the series.

Consider an angular opening with vertex on the positive real axis which includes the axis of reals in its interior. Suppose $a(z)$ has no singularities in this angular opening with the possible exception of

Presented to the Society, December 29, 1946; received by the editors May 29, 1946.

[1] By the term domain we mean an open connected set.

[2] See Dienes [1]. Numbers in brackers refer to the bibliography at the end of the paper.

the point at infinity. Let the sides of the angular opening make angles ψ_1 and ψ_2 with the axis of reals.

Our problem is to characterize the behavior of the function $f(z)$ in terms of the magnitudes of the angles ψ_1 and ψ_2 and the type of singularity $a(z)$ has at infinity.

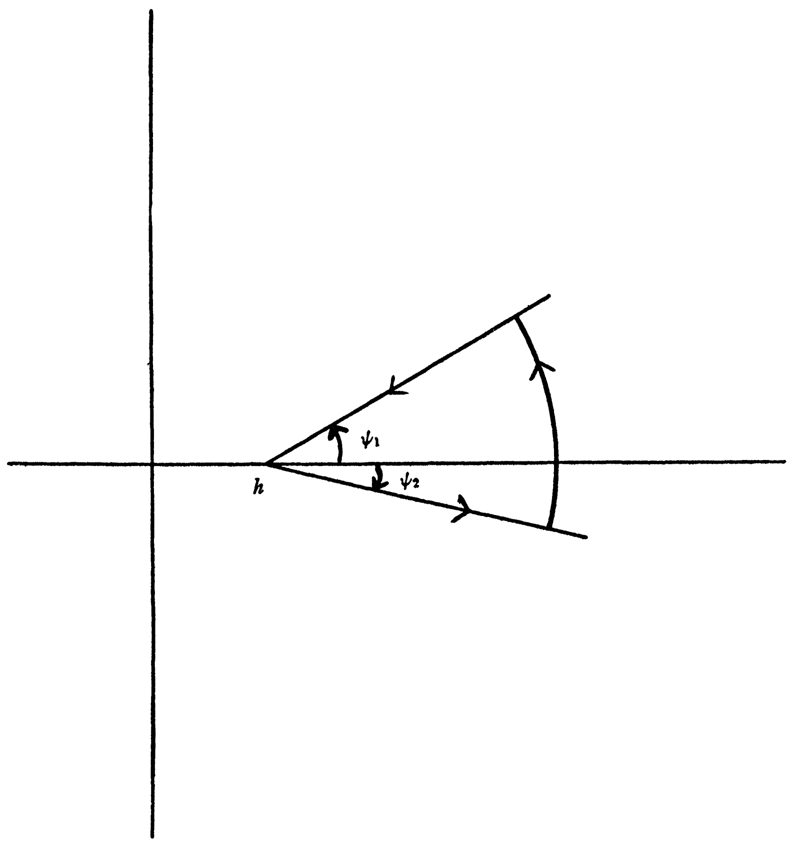

FIG. 1

We consider first the case where $a(z)$ may have a pole of order K at infinity and then the case in which infinity may be an essential singularity for $a(z)$.

2. $a(z)$ **may have a pole at infinity.** Suppose now that $a(z)$ is regular interior to and on the sides of the angular opening shown in Figure 1, except for at most a pole of order K at infinity. In Figure 1 let $l-1<h<l$ where l is a positive integer.

By the calculus of residues if $F(\omega)$ and $G(\omega)$ are uniform functions in a domain and if $G(\omega)$ has only simple zeros α_i in this domain, then (integration being understood in the positive sense)

$$\int_C \frac{F(\omega)}{G(\omega)}\, d\omega = \frac{F(\alpha_i)}{G'(\alpha_i)} \tag{1}$$

where C is a path enclosing α_i but no other zero of $G(\omega)$ and $G'(\alpha_i)$ is the derivative of $G(\omega)$ evaluated at $\omega=\alpha_i$.

Let $F(\omega)=a(\omega)z^\omega$ and $G(\omega)=\exp(2\pi i\omega)-1$.

For a given value of $z=r\exp(i\theta)$ we shall understand by z^ω

$$z^\omega = \exp[\omega(\log r + i\theta)], \qquad 0 \leqq \theta < 2\pi.$$

Here $\exp z$ is the principal value of e^z. This convention will be adhered to throughout the paper.

Then by (1)

$$\frac{1}{2\pi i}\int_C \frac{a(\omega)z^\omega}{\exp(2\pi i\omega) - 1}\, d\omega = a(n)z^n \tag{2}$$

where C is a path enclosing n and no other real integer. $a(\omega)$ is a function regular in the angular opening. This choice of $F(\omega)$ and $G(\omega)$ led to the well known theorem of LeRoy and Lindelöf [1] and is of course a well known method for the summation of certain series. The analysis of this paper follows lines similar to those in the analysis of Lindelöf.

Consider

$$\int_{C_{h,R}} \frac{a(\omega)z^\omega}{\exp(2\pi i\omega) - 1}\, d\omega$$

where $C_{h,R}$ is the path formed by the two sides of the angular opening, with vertex $h>0$, and the arc of a circle of radius R, where R is a positive integer. This is indicated in Figure 1.

By application of (2) it follows that

$$\frac{1}{2\pi i}\int_{C_{h,R}} \frac{a(\omega)z^\omega}{\exp(2\pi i\omega) - 1}\, d\omega = \sum_{n=l}^{R+l-1} a(n)z^n. \tag{3}$$

Denote the sides of the angular opening corresponding to ψ_1 and ψ_2 by I_1 and I_2 and the arc of a circle by C. Then

$$\begin{aligned}\int_{C_{h,R}} \frac{a(\omega)z^\omega}{\exp(2\pi i\omega) - 1}\, d\omega &= \int_{I_2} \frac{a(\omega)z^\omega}{\exp(2\pi i\omega) - 1}\, d\omega \\ &\quad + \int_C \frac{a(\omega)z^\omega}{\exp(2\pi i\omega) - 1}\, d\omega + \int_{I_1} \frac{a(\omega)z^\omega}{\exp(2\pi i\omega) - 1}\, d\omega.\end{aligned} \tag{4}$$

It will now be shown that, if we place certain restrictions on ψ_1, ψ_2 and z,

$$\int_C \frac{a(\omega)z^\omega}{\exp(2\pi i\omega) - 1}\, d\omega$$

converges uniformly to zero as R becomes infinite.

Let $z = r\exp(i\pi)$ and $\omega = h + R\exp(i\psi)$ with $-\pi < v \leqq \psi \leqq u < \pi$. By hypothesis there exists an R_0 such that in the angular opening with vertex $h > 0$

$$\left| a(h + R\exp(i\psi)) \right| < AR^K, \qquad R > R_0, \tag{5}$$

where A is a constant and K is a positive integer. On the arc C

$$\left| \frac{1}{\exp(2\pi i\omega) - 1} \right| < B, \tag{6}$$

where B is a constant. This follows from the fact that ω is bounded away from an integer by our choice of h, l and R.

Clearly

$$\left| \frac{1}{\exp(2\pi i\omega) - 1} \right| = \left| \frac{1}{1 - \exp(-2\pi i\omega)} \right| \left| \exp(-2\pi i\omega) \right|.$$

Hence on C,

$$\left| \frac{1}{\exp(2\pi i\omega) - 1} \right| < D\exp(2\pi R\sin\psi), \tag{7}$$

where D is a constant. This inequality proves useful for $-\pi < v \leqq \psi < 0$. Now

$$\left| \int_C \frac{a(\omega)z^\omega}{\exp(2\pi i\omega) - 1}\, d\omega \right| \leqq \int_C \frac{|a(\omega)|\,|z^\omega|}{|\exp(2\pi i\omega) - 1|} |d\omega|, \tag{8}$$

and

$$\begin{aligned} |z^\omega| &= \left| \exp\left[(h + R\cos\psi + iR\sin\psi)(\log r + i\pi)\right] \right| \\ &= r^h \exp\left[-R(\log r^{-1}\cos\psi + \pi\sin\psi)\right]. \end{aligned}$$

If $r < 1$, then

$$b(r, \psi) = \log r^{-1}\cos\psi + \pi\sin\psi > 0, \qquad 0 \leqq \psi \leqq \pi/2. \tag{9}$$

If $r > \exp(\pi\tan\psi)$, then

$$b_1(r, \psi) = \log r^{-1}\cos\psi + \pi\sin\psi > 0, \qquad \pi/2 \leqq \psi \leqq u < \pi. \tag{10}$$

Clearly $r > \exp(\pi\tan u)$ is sufficient for (10) to hold. We note here and also in (11) below, that the case $|\psi| = \pi/2$ is included. If $r<1$ then

$$(11) \qquad b_2(r, \psi) = \log r^{-1} \cos \psi - \pi \sin \psi > 0, \qquad -\pi/2 \leqq \psi \leqq 0.$$

If $r > \exp(-\pi \tan \psi)$, then

$$(12) \quad b_3(r, \psi) = \log r^{-1} \cos \psi - \pi \sin \psi > 0, \quad -\pi < v \leqq \psi \leqq -\pi/2.$$

Clearly $r > \exp(-\pi \tan v)$ is sufficient for (12) to hold.

We have seen that if $-\pi/2 \leqq \psi \leqq \pi/2$, $r<1$ implies $b(r, \psi) > 0$ and $b_2(r, \psi) > 0$. Denote, for $\alpha > 0$ but otherwise arbitrarily small, by r_1 the larger of $\exp(\pi \tan u) + \alpha$ and $\exp(-\pi \tan v) + \alpha$. Set $r_2 = 1 - \alpha'$, $\alpha' > 0$ but otherwise arbitrarily small. Then $r_1 \leqq r \leqq r_2$ implies $\pi/2 < u < \pi$ and $-\pi < v < -\pi/2$. We choose α and α' sufficiently small, so that $r_1 \leqq r \leqq r_2$ is an interval consisting of more than one point. Then, for a given α and α' and $r_1 \leqq r \leqq r_2$, (9), (10), (11), and (12) take on their minimum values on their respective intervals of definition. Let these be $b' > 0$, $b_1' > 0$, $b_2' > 0$ and $b_3' > 0$. Denote by $\beta > 0$ the smallest of these four values. Then by application of (5), (6) and (7) for $R > R_0$, (8) becomes

$$(13) \quad \begin{aligned} \left| \int_C \frac{a(\omega) z^\omega}{\exp(2\pi i \omega) - 1} d\omega \right| &\leqq T_0(A, B, D) R \int_v^u \exp(-\beta R) d\psi \\ &= T_0(A, B, D) R \exp(-\beta R)(u - v). \end{aligned}$$

Given an arbitrary $\epsilon > 0$, there exists an R_1 such that, for all $R > R_0$, R_1 and $r_1 \leqq r \leqq r_2$, the right-hand member of (13) is less than ϵ. That is

$$(14) \qquad \left| \int_C \frac{a(\omega) z^\omega}{\exp(2\pi i \omega) - 1} d\omega \right| < \epsilon, \qquad R > R_0, R_1; r_1 \leqq r \leqq r_2.$$

By (14) it follows that the integral along the arc converges uniformly to zero as R becomes infinite. Hence by (3) and (4) we may write

$$(15) \quad \sum_{n=l}^{\infty} a(n) z^n = \int_{I_2} \frac{a(\omega) z^\omega}{\exp(2\pi i \omega) - 1} d\omega + \int_{I_1} \frac{a(\omega) z^\omega}{\exp(2\pi i \omega) - 1} d\omega.$$

Let

$$J_{\psi_1}(z) = \int_{I_1} \frac{a(\omega) z^\omega}{\exp(2\pi i \omega) - 1} d\omega$$

and

$$J_{\psi_2}(z) = \int_{I_2} \frac{a(\omega) z^\omega}{\exp(2\pi i \omega) - 1} d\omega.$$

It will be shown that $J_{\psi_1}(z) + J_{\psi_2}(z)$ is regular for z in a domain which contains the interval $-r_2 \leqq z \leqq -r_1$ in its interior.

In this discussion four cases with regard to the angles ψ_1 and ψ_2 present themselves naturally.

A:	$-\pi/2 < \psi_2 < 0,$	$0 < \psi_1 < \pi/2$
B:	$-\pi/2 < \psi_2 < 0,$	$\psi_1 = \pi/2$
C:	$\psi_2 = -\pi/2,$	$\psi_1 = \pi/2$
D:	$-\pi < v \leqq \psi_2 \leqq -\pi/2,$	$\pi/2 < \psi_1 \leqq v < \pi.$

Consider case A. Here

$$\begin{aligned} J_{\psi_1}(z) &= \int_{I_1} \frac{a(\omega)z^\omega}{\exp(2\pi i\omega) - 1} d\omega \\ &= -\int_0^\infty \frac{a(\omega)\exp[(h + R\exp(i\psi_1))(\log r + i\theta)]}{\exp(2\pi i\omega) - 1} \exp(i\psi_1) dR. \end{aligned}$$

From (5) and (6) it follows that

$$(16) \qquad |J_{\psi_1}(z)| < ABr^h \int_0^\infty R^K \exp[-R(\log r^{-1}\cos\psi_1 + \theta\sin\psi_1)]dR.$$

If

$$(17) \qquad 0 < r < \exp(\theta\tan\psi_1), \qquad 0 \leqq \theta < 2\pi,$$

then $b(r, \theta, \psi_1) = \log r^{-1}\cos\psi_1 + \theta\sin\psi_1 > 0$.

Suppose $z = r\exp(i\theta)$ is in some closed domain[3] contained in the domain defined by (17). Denote by $b'(\psi_1) > 0$ the minimum assumed by $b(r, \theta, \psi_1)$ in this closed domain. Then (16) becomes

$$|J_{\psi_1}(z)| < ABr^h \int_0^\infty R^K \exp[-b'(\psi_1)R]dR < M,$$

where M is a constant.

Therefore $J_{\psi_1}(z)$ converges uniformly for z in any closed domain contained in (17). For ω on I_1 and z contained in the domain defined by (17) the integrand of $J_{\psi_1}(z)$ is continuous in ω and z. It follows from our definition of z^ω that for a fixed ω on I_1 the integrand is a regular function of z for z in any closed domain contained in (17). Then by well known theorems [2], $J_{\psi_1}(z)$ is regular for z in any closed domain contained in (17).

Consider

[3] D is said to be a closed domain if there exists a domain E with the property that $D = \bar{E}$. Here if E is a given set and E' its derived set then $\bar{E} = E + E'$.

$$J_{\psi_2}(z) = \int_{I_2} \frac{a(\omega)z^\omega}{\exp(2\pi i\omega) - 1} d\omega$$
$$= \int_0^\infty \frac{a(\omega) \exp [(h + R \exp(i\psi_2))(\log r + i\theta)]}{\exp(2\pi i\omega) - 1} \exp(i\psi_2) dR.$$

By (5) and (7) we may write

$$\begin{aligned} &| J_{\psi_2}(z) | \\ (18) \qquad &< ADr^h \int_0^\infty R^K \exp [- R \log r^{-1} \cos \psi_2 + (\theta - 2\pi) \sin \psi_2] dR. \end{aligned}$$

If

$$(19) \qquad 0 < r < \exp [(\theta - 2\pi) \tan \psi_2]$$

then $b(r, \theta, \psi_2) = \log r^{-1} \cos \psi_2 + (\theta - 2\pi) \sin \psi_2 > 0$.

Let $z = r \exp(i\theta)$ be in any closed domain contained in the domain defined by (19). Denote by $b'(\psi_2) > 0$ the minimum taken on by $b(r, \theta, \psi_2)$ in this closed domain; then (18) becomes

$$| J_{\psi_2}(z) | < ADr^h \int_0^\infty R^K \exp [- b'(\psi_2) R] dR < N,$$

where N is a constant.

It then follows by the same analysis employed in the case of $J_{\psi_1}(z)$ that $J_{\psi_2}(z)$ is regular in any closed domain contained in the domain defined by (19). The function $J_{\psi_1}(z) + J_{\psi_2}(z)$ will therefore be regular for $z = r \exp(i\theta)$ in any closed domain contained in the domain common to (17) and (19).

It has been shown (15) that

$$\sum_{n=l}^{\infty} a(n)z^n = J_{\psi_1}(z) + J_{\psi_2}(z)$$

where $z = r \exp(i\pi)$ and $r_1 \leqq r \leqq r_2$. But $J_{\psi_1}(z) + J_{\psi_2}(z)$ has been shown to be regular in a domain which includes the interval $-r_2 \leqq z \leqq -r_1$ in its interior. Hence $J_{\psi_1}(z) + J_{\psi_2}(z)$ provides the analytic continuation of the function defined by $\sum_{n=l}^{\infty} a(n)z^n$ to any closed domain contained in the domain common to (17) and (19). Since

$$f(z) = \sum_{n=0}^{l-1} a(n)z^n + \sum_{n=l}^{\infty} a(n)z^n,$$

it is evident that $J_{\psi_1}(z) + J_{\psi_2}(z) + \sum_{n=0}^{l-1} a(n)z^n$ provides the analytic continuation of $f(z)$ to the same domain. Hence in this case $f(z)$ will in general be regular in a domain bounded by two spirals as indicated in Figure 2.

Consider case B. By an analysis similar to that given in case A it is easily shown that $J_{\psi_1}(z)+J_{\psi_2}(z)$ is regular in any closed domain contained in the domain common to

$$0 < r < \exp\,[(\theta - 2\pi)\tan\psi_2], \qquad 0 \leq \theta < 2\pi, \tag{20}$$

and the whole complex plane excluding the segment 1 to $+\infty$. By an

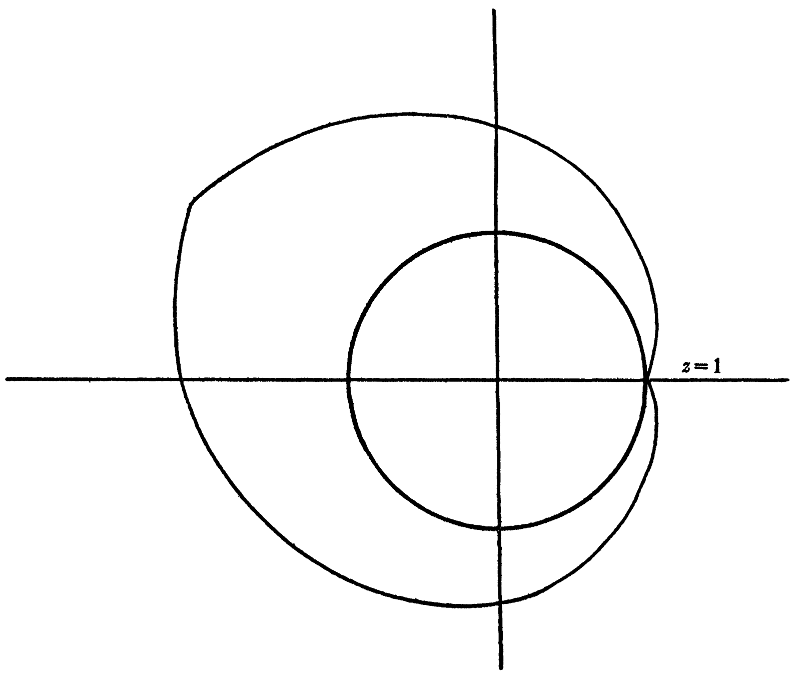

FIG. 2

argument similar to that employed in case A it is easily seen that $f(z)$ is regular in any closed domain contained in the domain common to (20) and the whole complex plane excluding the segment 1 to $+\infty$.

Case C. Here we have a very special case of the theorem of LeRoy and Lindelöf [1]. It could be shown by an analysis similar to the preceding that $f(z)$ is regular in any domain of the complex plane excluding the segment 1 to $+\infty$.

Finally we consider case D. By a method similar to that employed in case A it is simple to show that $J_{\psi_1}(z)$ converges uniformly for z in any closed bounded domain contained in the domain defined by

$$r > \exp(\theta \tan \psi_1), \qquad 0 \leqq \theta < 2\pi, \tag{21}$$

and that $J_{\psi_2}(z)$ converges uniformly for z in any closed bounded domain contained in the domain defined by

$$r > \exp[(\theta - 2\pi)\tan \psi_2], \qquad 0 \leqq \theta < 2\pi. \tag{22}$$

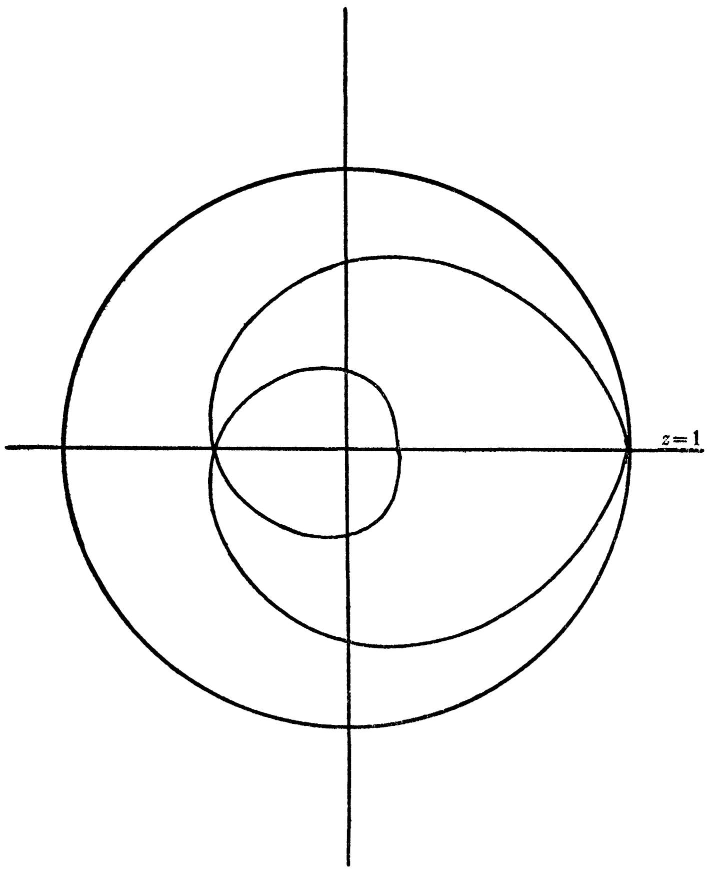

FIG. 3

Now the domain common to (21) and (22) contains the segment $-r_2 \leqq z \leqq -r_1$ in its interior. This is easily seen by setting $\theta = \pi$ in (21) and (22) and noting that r_1 is the larger of $\exp(\pi \tan u) + \alpha$ and $\exp(\pi \tan v) + \alpha$, $\alpha > 0$. Hence by arguments similar to those used in case A

$$J_{\psi_1}(z) + J_{\psi_2}(z) + \sum_{n=0}^{l-1} a(n) z^n$$

provides the analytic continuation of $f(z)$ to any closed bounded domain contained in the domain common to (21) and (22). For the case $\psi_1=\psi_2$ the spirals (21) and (22) are indicated in Figure 3. In calculating $J_{\psi_1}(z)$ and $J_{\psi_2}(z)$ for a given $z=r\exp(i\theta)$ we recall that by our convention

$$z^\omega = \exp[\omega(\log r + i\theta)], \qquad 0 \leqq \theta < 2\pi.$$

Hence we have the following theorem.

THEOREM 1. *Let $f(z)=\sum_{n=0}^\infty a_n z^n$ with radius of convergence unity. Let the coefficients a_n be the values taken on by an analytic function $a(z)$ at $z=0, 1, 2, \cdots$. Suppose $a(z)$ is regular with the possible exception of a pole of order K at infinity, in an angle with vertex $h>0$ (non-integral) on the axis of reals and including the axis of positive reals in its interior. Let the sides of this angle make angles ψ_1 and ψ_2 with the axis of reals. Then, if $\gamma>0$ but otherwise arbitrarily small, and*

A: $$0<\psi_1<\pi/2, \qquad -\pi/2<\psi_2<0,$$

$f(z)$ is regular in the domain common to

$$r \leqq \exp[\theta \tan\psi_1] - \gamma, \qquad 0 \leqq \theta < 2\pi,$$

and

$$r \leqq \exp[(\theta-2\pi)\tan\psi_2] - \gamma, \qquad 0 \leqq \theta < 2\pi.$$

If

B: $$\psi_1=\pi/2, \qquad -\pi/2<\psi_2<0,$$

$f(z)$ is regular in any closed domain common to

$$r \leqq \exp[(\theta-2\pi)\tan\psi_2] - \gamma, \qquad 0 \leqq \theta < 2\pi,$$

and the whole complex plane excluding the segment 1 *to* $+\infty$. *If*

C: $$\psi_1=\pi/2, \qquad \psi_2=-\pi/2,$$

$f(z)$ is regular in any closed domain of the complex plane excluding the segment 1 *to* $+\infty$. *If*

D: $$\pi/2<\psi_1\leqq u<\pi, \qquad -\pi<v\leqq\psi_2<-\pi/2,$$

$f(z)$ is regular in any bounded domain common to

$$r \geqq \exp(\theta\tan\psi_1) + \gamma, \qquad 0 \leqq \theta < 2\pi,$$

and

$$r \geqq \exp [(\theta - 2\pi) \tan \psi_2] + \gamma, \qquad 0 \leqq \theta < 2\pi.$$

It is clear[4] that the theorem above is still true even though $a(z)$ does not have a pole at infinity. All that is necessary is that $a(z)$ be single-valued in the angle and that $|a(h+R\exp(i\psi))| < AR^K$, $R > R_0$.

3. $a(z)$ **may have an essential singularity at infinity.** Suppose $a(z)$ is regular interior to and on the sides of the angular opening of Figure 1. Suppose there exists an R_0 such that for $R > R_0$ and $z = h + R\exp(i\psi)$ in this angle

$$|a(h + R\exp(i\psi))| < \exp(\delta R). \tag{23}$$

In order to simplify the work to follow suppose $\delta \leqq \pi - d$, $d > 0$.

It will be shown that, if we place certain restrictions on ψ_1, ψ_2 and z,

$$\int_C \frac{a(\omega)z^\omega}{\exp(2\pi i\omega) - 1} d\omega \tag{24}$$

converges uniformly to zero as R becomes infinite. Let $z = r\exp(i\pi)$ and $\omega = h + R\exp(i\psi)$ with $-\pi/2 \leqq \psi \leqq \pi/2$.

If $r < \exp[\pi \tan \psi - \delta \sec \psi]$, then

$$b_1(r, \psi) = \log r^{-1} \cos \psi + \pi \sin \psi - \delta > 0. \tag{25}$$

If $r < \exp[-\pi \tan \psi - \delta \sec \psi]$, then

$$b_2(r, \psi) = \log r^{-1} \cos \psi - \pi \sin \psi - \delta > 0. \tag{26}$$

Set $g(\psi) = \pi \tan \psi - \delta \sec \psi$ where $0 \leqq \psi \leqq \pi/2$ and $\delta \leqq \pi - d$, $d > 0$. Now $g'(\psi) = \sec \psi(\pi \sec \psi - \delta \tan \psi)$. Hence for $\delta \leqq \pi - d$ and $0 \leqq \psi < \pi/2$, $g'(\psi)$ is positive.

As ψ approaches $\pi/2$, $g(\psi)$ approaches $+\infty$. Hence the minimum value of $g(\psi)$ on the interval $0 \leqq \psi \leqq \pi/2$ is $g(0)$, that is $-\delta$. In a similar manner we see that the minimum of $-\pi \tan \psi - \delta \sec \psi$ on the interval $-\pi/2 \leqq \psi \leqq 0$ is $-\delta$.

Hence if $r < \exp[-\delta - p]$ where $p > 0$ but otherwise arbitrarily small, (25) holds for $0 \leqq \psi \leqq \pi/2$ and (26) holds for $-\pi/2 \leqq \psi \leqq 0$.

Denote by $b_1' > 0$ the minimum assumed by $b_1(r, \psi)$ on the interval $0 \leqq \psi \leqq \pi/2$ and by $b_2' > 0$ the minimum assumed by $b_2(r, \psi)$ on the interval $-\pi/2 \leqq \psi \leqq 0$ where $0 \leqq r \leqq \exp[-\delta - p]$, $p > 0$. Let $b_0 > 0$ be the smaller of $b_1' > 0$ and $b_2' > 0$.

Then from (6), (7) and (23) we have

---

[4] The author is indebted to the referee for this observation.

$$(27)\quad \left| \int_C \frac{a(\omega)z^\omega d\omega}{\exp(2\pi i\omega) - 1} \right| \leqq Rr^h T_1(B, D) \int_{-\pi/2}^{\pi/2} \exp[-b_0 R] d\psi$$
$$= Rr^h T_1(B, D) \exp[-b_0 R]\pi.$$

Given an $\epsilon > 0$ we can choose an R_1 such that for $R > R_0$, R_1 and $0 \leqq r \leqq \exp[-\delta - p]$, $p > 0$, the quantity on the right of (27) is less than ϵ. That is, (24) converges uniformly to zero as R becomes infinite.

Let $\pi/2 \leqq \psi \leqq u < \pi$. If $r > \exp[\pi \tan \psi - \delta \sec \psi]$ then

$$(28)\quad b_1(r, \psi) = \log r^{-1} \cos \psi + \pi \sin \psi - \delta > 0.$$

Let $-\pi < v \leqq \psi \leqq -\pi/2$. If $r > \exp[-\pi \tan \psi - \delta \sec \psi]$ then

$$(29)\quad b_2(r, \psi) = \log r^{-1} \cos \psi + \pi \sin \psi - \delta > 0.$$

Denote, for a given $q > 0$ but otherwise arbitrarily small, by F the smaller of the numbers $\pi \sin u - q$ and $\pi \sin v - q$. Let $\delta \leqq F$. The maximum for $\pi/2 \leqq \psi \leqq u < \pi$ of $\pi \tan \psi - \delta \sec \psi$ is $\pi \tan \psi - F \sec \psi$. The maximum for $-\pi < v \leqq \psi \leqq -\pi/2$ of $-\pi \tan \psi - \delta \sec \psi$ is $\pi \tan v - F \sec v$. Denote by E the larger of $\pi \tan u - F \sec u + t$ and $\pi \tan v - F \sec v + t$ where $t > 0$ but sufficiently small that $E < 0$. If $r > \exp(E)$ then (28) and (29) hold. Suppose now in addition that u and v have the property that

$$(30)\quad \cos u > -\frac{\pi \sin u - F}{\delta + p + t}$$

and

$$(31)\quad \cos v > -\frac{\pi \sin v - F}{\delta + p + t}\cdot$$

Here p and t are positive but otherwise may be chosen arbitrarily small. Let $z = r \exp(i\pi)$ with

$$(32)\quad \exp E \leqq r \leqq \exp(-\delta - p).$$

That there is an interval consisting of more than one point satisfying (32) follows from the restrictions (30) and (31) placed on u and v. For if u and v satisfy (30) and (31) then $E < -\delta - p$. Denote by $b_1'' > 0$ the minimum assumed by $b_1(r, \psi)$ on $\pi/2 \leqq \psi \leqq u < \pi$ and by $b_2'' > 0$ the minimum assumed by $b_2(r, \psi)$ on $-\pi < v \leqq \psi \leqq -\pi/2$ where r satisfies (32). Denote by $b_0'' > 0$ the smaller of b_1'' and b_2''. Let $b^* > 0$ be the smaller of b_0 and b_0''. Then by (6), (7) and (23) if $-\pi < v \leqq \psi \leqq u < \pi$, $\delta \leqq F$ and $\exp E \leqq r \leqq \exp(-\delta - p)$, where u and v satisfy (30) and (31) we have

$$(33)\qquad \left| \int_C \frac{a(\omega)z^\omega}{\exp(2\pi i\omega)-1}\,d\omega \right| \leqq Rr^h T_2(B, D)\int_v^u \exp(-b^*R)d\psi \leqq Rr^h T_2(B, D)\exp(-b^*R)(u-v).$$

Given an $\epsilon>0$ there exists an R_1 such that for $R>R_0$, R_1 and $\exp E\leqq r\leqq\exp(-\delta-p)$ the quantity on the right of (33) is less than ϵ. Hence as R becomes infinite we again obtain (15) since (24) converges uniformly to zero.

It is now possible to consider again the four cases of §2; however, for brevity we shall consider only those corresponding to A and D. Let us denote these by A′ and D′.

A′: $\quad 0<\psi_1<\pi/2, \qquad -\pi/2<\psi_2<0,$

D′: $\quad \pi/2<\psi_1\leqq u, \qquad v\leqq\psi_2<-\pi/2.$

Case A′. Suppose that $\delta\leqq\pi-d$, $d>0$. It will be shown that $J_{\psi_1}(z)+J_{\psi_2}(z)$ is regular for z in a domain which includes all or part of the segment $-\exp[-\delta-p]\leqq z<0$ in its interior. By a method similar to that employed in §2 we could show that for $J_{\psi_1}(z)$ to converge for a fixed $z=r\exp(i\theta)$ it is sufficient that

$$(34)\qquad 0<r<\exp\left[\theta\tan\psi_1-\delta\sec\psi_1\right],\qquad 0\leqq\theta<2\pi.$$

It follows then that $J_{\psi_1}(z)$ will converge uniformly for z in any closed domain contained in the domain defined by (34). Hence by reasoning similar to that employed in §2, it is easily seen that $J_{\psi_1}(z)$ is regular for z in any closed domain contained in (34).

Consider $J_{\psi_2}(z)$. It is easily shown that it converges for a fixed $z=r\exp(i\theta)$ contained in the domain defined by

$$(35)\qquad 0<r<\exp\left[(\theta-2\pi)\tan\psi_2-\delta\sec\psi_2\right],\quad 0\leqq\theta<2\pi.$$

If then $z=r\exp(i\theta)$ is in any closed domain contained in the domain defined by (35), $J_{\psi_2}(z)$ converges uniformly and hence represents a regular function. The function $J_{\psi_1}(z)+J_{\psi_2}(z)$ will therefore be regular in any closed domain contained in the domain common to (34) and (35). We have seen that if $z=r\exp(i\pi)$ with $\exp E\leqq r\leqq\exp[-\delta-p]$, $p>0$, that

$$\sum_{n=l}^{\infty} a(n)z^n = J_{\psi_1}(z)+J_{\psi_2}(z).$$

But $J_{\psi_1}(z)+J_{\psi_2}(z)$ has been shown to be regular in a domain which includes all or part of the interval $\exp E\leqq r\leqq\exp[-\delta-p]$ in its interior. Therefore $J_{\psi_1}(z)+J_{\psi_2}(z)+\sum_{n=0}^{l-1}a(n)z^n$ provides the analytic continuation of $f(z)$ to any closed domain contained in the domain common to (34) and (35). Hence in this case $f(z)$ will be regular in a domain bounded by two spirals as indicated in Figure 4.

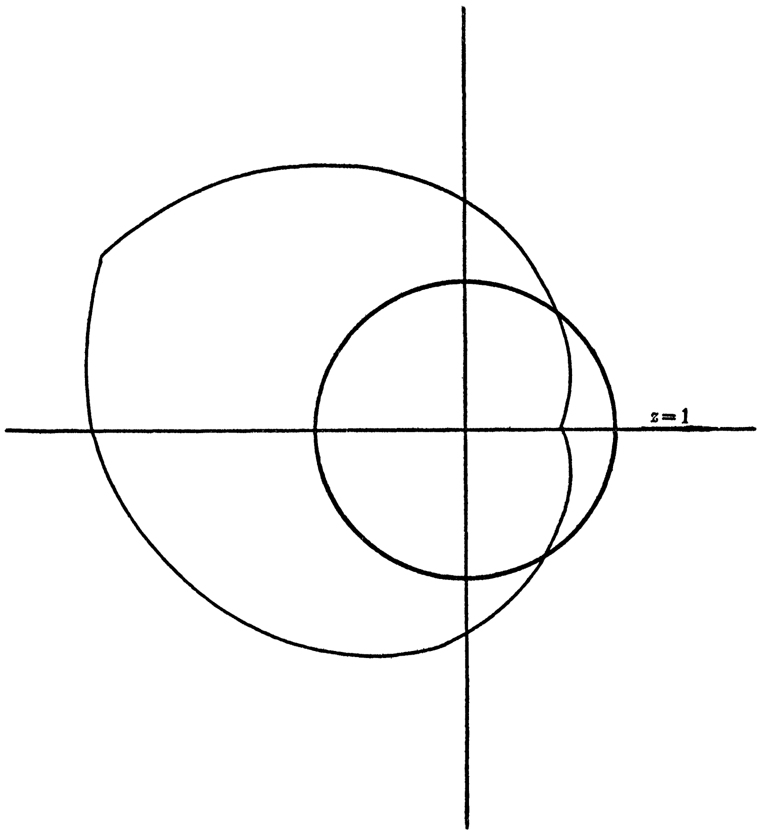

FIG. 4

Consider case D′. Suppose $\delta \leqq F$ and that v and u satisfy (30) and (31). Then for $\exp E \leqq r \leqq \exp(-\delta - p)$ it follows from (33) that (15) holds. It is easily seen that $J_{\psi_1}(z)$ will converge for a fixed $z = r \exp(i\theta)$ if $\log r^{-1} \cos \psi_1 + \theta \sin \psi_1 - \delta > 0$. This will be the case if

$$r > \exp\left[\theta \tan \psi_1 - \delta \sec \psi_1\right], \qquad 0 \leqq \theta < 2\pi. \tag{36}$$

It is evident that $J_{\psi_1}(z)$ will converge uniformly for $z = r \exp(i\theta)$ in any closed bounded domain contained in the domain defined by (36). In order that $J_{\psi_2}(z)$ converge for a fixed $z = r \exp(i\theta)$ it is sufficient that $\log r^{-1} \cos \psi_2 + (\theta - 2\pi) \sin \psi_2 - \delta > 0$. That is,

$$r > \exp\left[(\theta - 2\pi) \tan \psi_2 - \delta \sec \psi_2\right], \qquad 0 \leqq \theta < 2\pi. \tag{37}$$

Then $J_{\psi_2}(z)$ will converge uniformly for $z=re^{i\theta}$ in any closed bounded domain contained in the domain defined by (37). Hence

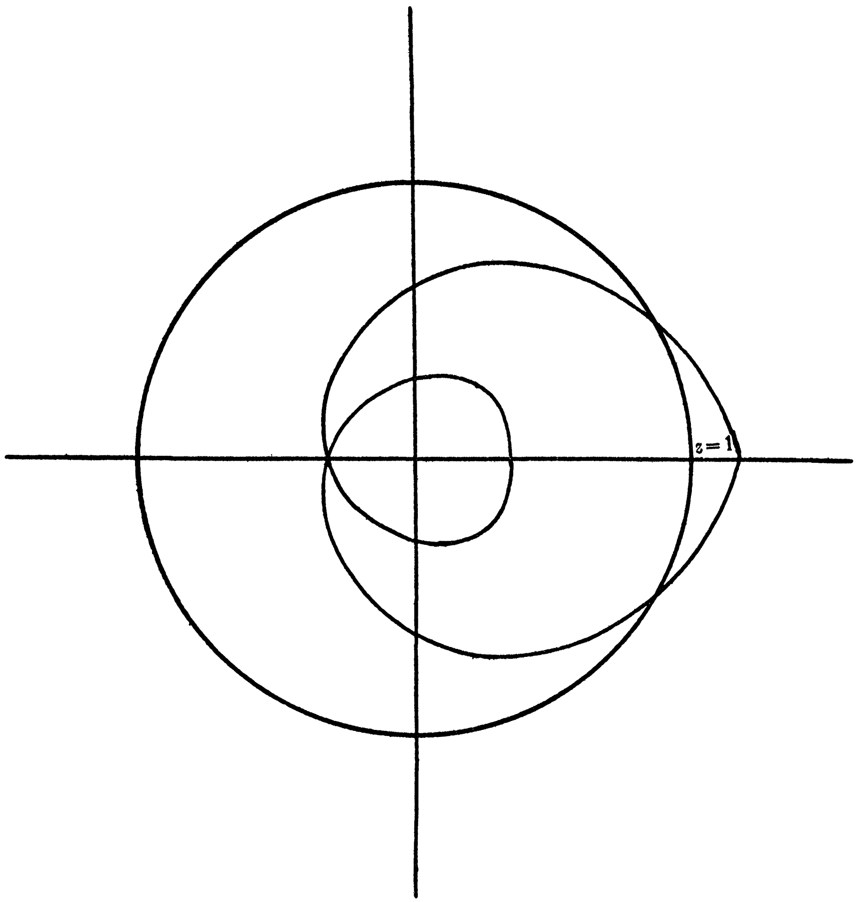

FIG. 5

$J_{\psi_1}(z)+J_{\psi_2}(z)$ will converge uniformly for $z=re^{i\theta}$ in any closed bounded domain contained in the domain common to (36) and (37). It is evident that the region common to (36) and (37) contains the interval $-\exp(-\delta-p)\leqq z\leqq -\exp E$ in its interior. For E is by definition the larger of $\pi\tan u-F\sec u+t$ and $\pi\tan v-F\sec v+t$, $t>0$, and if we set $\theta=\pi$ in (36) and (37) it is clear that both exponents are smaller than E. Therefore $J_{\psi_1}(z)+J_{\psi_2}(z)+\sum_{n=0}^{l-1}a(n)z^n$ provides the analytic continuation of $f(z)$ to any closed bounded domain contained in the domain common to (36) and (37). For the case $\psi_1=\psi_2$ this will be such a domain as indicated in Figure 5.

Hence we have the following theorem.

THEOREM 2. *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ with radius of convergence unity. Let the coefficients a_n be the values taken on by an analytic function $a(z)$ at $z=0, 1, 2, \cdots$. Suppose $a(z)$ is regular with the possible exception of an essential singularity at infinity in an angle with vertex $h>0$ (non-integral) on the real axis, including the axis of positive reals in its interior. Let the sides of this angle make angles ψ_1 and ψ_2 with the axis of positive reals. Then if, for $z=h+R\exp(i\psi)$ in this angular opening, $a(z)$ satisfies the inequality*

$$|a(h+R\exp(i\psi))| < \exp(\delta R), \qquad R > R_0,$$

where $\delta\leqq\pi-d$, $d>0$, and

A′: $$0 < \psi_1 < \pi/2, \qquad -\pi/2 < \psi_2 < 0,$$

$f(z)$ is regular in the domain common to

$$r \leqq \exp[\theta\tan\psi_1 - \delta\sec\psi_1] - \gamma, \qquad \gamma > 0$$

and

$$r \leqq \exp[(\theta-2\pi)\tan\psi_2 - \delta\sec\psi_2] - \gamma, \qquad \gamma > 0,$$

for $0\leqq\theta<2\pi$.

For a given $q>0$ but otherwise arbitrarily small let F be the smaller of the numbers $\pi\sin u-q$ and $\pi\sin v-q$. Suppose u and v may be chosen such that for a given p and t positive but otherwise arbitrarily small

$$\cos u > -\frac{\pi\sin u - F}{\delta+p+t}$$

and

$$\cos v > -\frac{\pi\sin v - F}{\delta+p+t}\cdot$$

Then if $\delta\leqq F$ and

D′: $$\pi/2 < \psi_1 \leqq u < \pi, \qquad -\pi < v \leqq \psi_2 < -\pi/2,$$

$f(z)$ is regular in any bounded domain common to

$$r \geqq \exp[\theta\tan\psi_1 - \delta\sec\psi_1] + \gamma, \qquad \gamma > 0,$$

and

$$r \geqq \exp[(\theta-2\pi)\tan\psi_2 - \delta\sec\psi_2] + \gamma, \qquad \gamma > 0,$$

where $0\leqq\theta<2\pi$.

4. Conclusions. We note first that if, in Theorem 1, $\psi_1>0$ and $\psi_2<0$ but otherwise arbitrarily small, that $f(z)$ has $z=1$ as its only singularity on the circle of convergence.

In part D of Theorem 1 if both ψ_1 and ψ_2 are greater than 90° in magnitude, that is, the sector of regularity is greater than 180°, we have the rather remarkable result that $z=1$ is the only singularity of $f(z)$ in the finite plane. Thus for example the function defined by the series

$$\sum_{n=0}^{\infty} \frac{z^n}{(n+\alpha)^\beta},$$

where β is an integer and α is not equal to zero or a negative integer but otherwise arbitrary, has the point $z=1$ as its only singularity in the finite plane.

It is of course clear that we may use the results obtained in a different manner, that is, if $f(z)$ has a singular point on the circle of convergence other than $z=1$ then $a(z)$ cannot be analytic in an angular opening including the axis of positive reals in its interior with at most a pole of finite order at infinity.

If the inequality for $a(z)$ in Theorem 2 is satisfied for every $\delta>0$ however small, then under the condition of case D′ of Theorem 2, $z=1$ is the only singularity in the finite plane. This result[5] is analogous to the following theorem due to Faber [1].

THEOREM (FABER). *If $g(z)$ is an integral function such that $|g(re^{i\theta})|<e^{\epsilon r}$ for an arbitrary positive ϵ and $r>r'$, the function $f(z)$ defined by $\sum_{n=0}^{\infty} g(n)z^n$ and its analytic continuation has the point* 1 *as its only singular point.*

We observe now that the bounding curves

$$r < \exp\left[\theta \tan \psi_1 - \delta \sec \psi_1\right], \qquad 0 \leqq \theta < 2\pi,$$

and

$$r < \exp\left[(\theta - 2\pi) \tan \psi_2 - \delta \sec \psi_2\right], \quad 0 \leqq \theta < 2\pi,$$

of (34) and (35) cut the unit circle at the points $\exp\left[i\delta \csc \psi_1\right]$ and $\exp\left[i(2\pi+\delta \csc \psi_2)\right]$. If now, in addition to the requirements of part A of Theorem 2, ψ_1, ψ_2 and δ satisfy the inequality

$$2\pi > \delta(\csc \psi_1 - \csc \psi_2)$$

it is easily seen that the region common to (34) and (35) will extend beyond the unit circle. We then have the following theorem.

[5] The author is indebted to the referee for pointing out the analogy.

THEOREM 3. *If the conditions of Theorem 2 part A are satisfied and if in addition the quantities ψ_1, ψ_2 and δ satisfy the inequality*

$$2\pi > \delta(\csc \psi_1 - \csc \psi_2)$$

then the circle of convergence is not a cut for the function.

BIBLIOGRAPHY

1. P. Dienes, *The Taylor series*, Oxford, 1931.
2. E. C. Titchmarsh, *The theory of functions*, Oxford University Press, 1939, pp. 99–100.

LEHIGH UNIVERSITY

A NOTE ON THE HILBERT TRANSFORM

LYNN H. LOOMIS

The Hilbert transform of $f(t)$, $-\infty < t < \infty$, is $1/\pi$ times the Cauchy principal value

$$\bar{f}(x) = P\int_{-\infty}^{\infty} \frac{f(t)}{t-x}\,dt = \lim_{\delta \to 0+} \int_{\delta}^{\infty} \frac{f(x+t) - f(x-t)}{t}\,dt.$$

If $f(t) \in L_p$, $p > 1$, then $\bar{f}(x) \in L_p$, and a considerable literature is devoted to studying the relationship of such pairs of "conjugate" functions to the theory of functions analytic in a half-plane. More to the point of the present note is a series of papers studying the Hilbert transform along strictly real variable lines ([2, 3]; further bibliography in [2]).[1]

Much less is known about $\bar{f}(x)$ when $f(t) \in L_1$. Plessner found by applying complex variable methods to the theory of Fourier series that if $f(t) \in L_1$ then $\bar{f}(x)$ exists almost everywhere (see [1, p. 145]). Besicovitch [4] proved Plessner's result using only the theory of sets, starting from his own previous real variable investigation of the L_2 transform case. S. Pollard [5] showed how Besicovitch's proof could be extended to prove the existence a.e. of the principal value of the Stieltjes integral

$$\bar{f}(x) = P\int_{-\infty}^{\infty} \frac{dF(t)}{t-x},$$

Received by the editors April 11, 1946.

[1] Numbers in brackets refer to the bibliography at the end of the paper.