play than the classical checker game. The checkers family can also be extended to spaces of higher dimensions. (Received July 12, 1946.)

Statistics and Probability

326. H. W. Becker: *Stirling's numbers of the third kind.*

These are defined by ${}_{c\pm}^{x}S_{\pm r+1}=(x\pm c\pm r)\cdot{}_{c\pm}^{x}S_{\pm r}+{}_{c-1\pm}S_{\pm r}$, in which the weighting coefficient combines the characteristics of the recurrences for the Stirling's numbers of the first and second kinds. However, tables of the four varieties $\pm\pm$ are not only matrix products of certain tables of the first two kinds, but also unexpectedly simple vector multiples of the table of binomial coefficients. Summed over c, these are expressible by polynomials which are Chrystal-Jordan factorials of Stirling's polynomials of the second kind, and have the remarkable property of a triple recurrence. For example, ${}_{+}^{x}S_{-r+1}=(1+(x))^{r+1}=({}_{+}S+x)_{r+1}={}_{+}^{x}S_{-r}+x\cdot{}^{x-1}_{+}S_{-r}={}^{x-1}_{+}S_{-r+1}+(r+1)\cdot{}^{x-1}_{+}S_{-r}={}^{x+1}_{+}S_{-r}+(x-r)\cdot{}_{+}^{x}S_{-r}$. The combinatorial interpretations are rather intricate in terms of either permutations or rhyme schemes. (Received July 15, 1946.)

327. David Blackwell: *Conditional expectation and unbiased sequential estimation.*

It is shown that $E[f(x_\alpha)E_\alpha y]=E(fy)$ whenever $E(fy)$ is finite, and that $\sigma^2(E_\alpha y)\leqq\sigma^2(y)$, with equality holding only if $E_\alpha y=y$, where $E_\alpha y$ denotes the conditional expectation of y with respect to the family of chance variables x_α. These results imply that whenever there is a sufficient statistic u and an unbiased estimate t, not a function of u only, for a parameter p, the function $E_u t$, which is a function of u only, is an unbiased estimate for p with variance smaller than that of t. A sequential unbiased estimate for a parameter is obtained, such that when the sequential test terminates after i observations, the estimate is a function of a sufficient statistic for the parameter with respect to these observations. A special case of this estimate is that obtained by Girshick, Mosteller, and Savage (Ann. Math. Statist. vol. 17 (1946) pp. 13–23) for the parameter of a binomial distribution. (Received July 5, 1946.)

328. Paul Boschan: *The consolidated Doolittle technique.*

The quadratic matrix notation is interpreted as a segment in a sequence of matrices wherein each successor matrix is augmented by a bordering row and column. Extension theorems based on this idea date back into the last century. The step from the original concept to one of higher order is also fruitful in discussing inverse matrices, specifically the inverse of a symmetric matrix. The symmetry of the matrix of normal equations for a set of multiple regression-coefficients is restored by adding the transpose of the column on the right side of the equations, that is, the co-variances with the dependent variable and the variance of the dependent variable itself. The inverse of this matrix can be constructed as a partial sum over a series of matrices. Each individual element of this series is in itself meaningful. The solution for the set of multiple regression coefficients relating the kth variable to the preceding $(k-1)$ variables is a column matrix. The product of this matrix with its transpose expressed in terms of the residual variance forms the kth term in the matrix series. The summation of the first n products yields the inverse matrix. This characteristic of the inverse can be used to great advantage in the standardization of elementary computational steps. (Received July 17, 1946.)

329. P. G. Hoel: *The efficiency of the mean moving range.*

The statistic $w=\sum_1^{n-1}|x_{i+1}-x_i|\ \pi^{1/2}/2(n-1)$ is studied as an estimate of σ for a normal variable subject to trend effects. It is shown that the efficiency of w compares favorably with that of the mean square successive difference, δ^2. The proof that w, and also δ^2, is asymptotically normally distributed is made to depend upon a general result that can be derived from a theorem of S. Bernstein on dependent variables. (Received July 13, 1946.)

330. Leo Katz: *On the class of functions defined by the difference equation* $(x+1)f(x+1)=(a+bx)f(x)$.

The difference equation defines only three discrete functions: the binomial, the Poisson and the Pascal functions; the first and third have one parameter (N) slightly generalized. It is shown that the Pascal function with this generalization is identical with the Polya-Eggenburgher distribution, which is a very useful form of the compound Poisson law and has been used to explain probability situations involving contagion. Areas for all functions in the class are given in terms of existing tables of the incomplete- and B-functions. Observed distributions are fitted by two moments. As Carver (*Handbook of mathematical statistics*) pointed out, the advantages of fitting by difference equations are many; not the least is the fact that it is unnecessary to discriminate among the various functions in fitting an observed distribution. The problem of discrimination, posed by Frisch (Metron vol. 10) and others, may be resolved in terms of the sampling distribution of variances for the Poisson function, since the three functions correspond to situations where the variance is less than, equal to, or greater than the mean, respectively. (Received July 13, 1946.)

331. B. F. Kimball: *Some basic theorems for developing tests of fit for the case of the non-parametric probability distribution function.*

Given a universe with c.d.f. $P[X\leqq x]=F(x)$, consider a random sample of n values x_i which have been ordered so that $x_i\leqq x_{i+1}$. The successive differences of the true c.d.f. values at $X=x_i$ are denoted by u_i. Thus $u_1=F(x_1)$, $u_i=F(x_i)-F(x_{i-1})$, $2\leqq i\leqq n$, $u_{n+1}=1-F(x_n)$. Theorem 1: The product power moments $E(u_r^p u_s^q u_t^w \cdots)$ for any or all different indices from 1 to $n+1$, where the powers are real numbers greater than -1, are given by $E(u_r^p u_s^q u_t^w \cdots)=\Gamma(n+1)\Gamma(p+1)\Gamma(q+1)\Gamma(w+1)\cdots/\Gamma(n+1+p+q+w+\cdots)$. Theorem 2: Given a test function of u_i, $Y=\sum_m u_i^p$, where p is a real positive number, and the sum is for m indices chosen at random on the range 1 to $n+1$, let $\bar{Y}$ and σ^2 denote the mean and variance of this test function. Establish a convention for increasing the indices included in the above sum for increasing m as n increases, such that $[m/(n+1)]=$const., to nearest multiple of $1/(n+1)$. It is proved that the asymptotic distribution of $(Y-\bar{Y})/\sigma$ for increasing n, subject to the above condition, is the normal distribution with zero mean and unit variance, except in the trivial case $m=n+1$, $p=1$. (Received July 16, 1946.)

332. Herman Rubin: *Asymptotic distribution of moments from a system of linear stochastic difference equations.*

Let $\sum_{\tau=0}^{\infty}B_\tau y'_{t-\tau}+\Gamma z'_t=u'_t$ $(t=1, 2, \cdots)$ be a complete system of linear stochastic difference equations determining y_{ti} (the coordinates of y_t), $t>0$, in terms of y_{ti}, $t\leqq 0$, and z_{tk} (the coordinates of z_t), which are assumed to be fixed variates, and the random variables u_{ti} (the coordinates of u_t). Such a system is called stable if for every bounded set of fixed variates, and $E(u_t' u_t)$ uniformly bounded, $E(y_t' y_t)$ is uniformly bounded. This condition is shown to be equivalent to $\sum |h_{ij\tau}|$ finite, where $y_t' = \sum_{\tau=0}^{\infty} H_\tau(u_{t-\tau}' - \Gamma z_{t-\tau}') + \sum_{\nu=0} J_{t,\nu} y_{-\nu}'$ is the solution of the above difference equation. Let Q_t be an infinite quadratic form in $y_{t-\tau,i}$ and $z_{t-\nu,k}$ $(\tau, \nu=0, 1, \cdots)$ with coefficients depending only on i, k, τ, and ν. Such a quadratic form is called convergent if the sum of the absolute values of the coefficients is finite. It is shown under fairly general conditions that the mean of a convergent quadratic form is asymptotically normally distributed with variance $O(1/T)$. (Received July 8, 1946.)

333. F. E. Satterthwaite: *Retention of decimal places in matrix calculations.*

The accumulation of errors in matrix calculations has been studied by the author and others for special types of matrices and for special methods of calculation. In the present paper, error formulae are developed for the standard Doolittle and Waugh-Dwyer compact routines. These formulae do not place any restrictions on the matrices involved and do not require any extra calculations or initial approximations. Simple rules are developed which give for each step in the calculations the number of decimal places which must be retained. These rules are efficient in the sense that the retention of fewer places will, except for good fortune in balancing of errors, lead to results less accurate than those specified. The rules also assist in choosing that arrangement of the calculations which will lead to the smallest average number of significant figures which must be retained for the calculation as a whole. (Received July 15, 1946.)

334. J. W. Tukey: *Sampling from contaminated distributions.* Preliminary report.

A contaminated distribution is a nearly normal distribution in which extreme observations are more frequent than in a normal distribution. By studying the bias and variability of several measures of dispersion when applied to samples from particular one-parameter families of contaminated distributions it is shown that: (i) for nearly normal distributions, the mean deviation is often better than the standard deviation; (ii) small changes in the underlying distribution may increase the sampling variance of the standard deviation by a factor of three. This suggests that, in a broad class of cases, the mean deviation is safer than the standard deviation when a single dispersion is estimated from a set of data. This conclusion need not apply in an analysis of variance situation. (Received July 13, 1946.)

335. Jacob Wolfowitz: *Confidence limits for the fraction of a normal population which lies between two given limits.*

Let $x_1, \cdots, x_N$ be N independent observations from a normal population with mean μ and variance σ^2, both unknown. Let $N\bar{x} = \sum x_i$ and $(N-1)s^2 = \sum (x_i - \bar{x})^2$ define $\bar{x}$ and s^2. Let L_1 and L_2 be given constants with $L_1 < L_2$, and let $\gamma = ((2\pi)^{1/2}\sigma)^{-1} \int_{L1}^{L} \exp(-1/2)\{(y-\mu)/\sigma\}^2 dy$. By a lower confidence limit on γ with confidence coefficient α is meant a function $D(x_1, \cdots, x_N)$ such that the probability is α that $D \leqq \gamma$. Since $\bar{x}$ and s^2 are sufficient estimates of μ and σ^2 the restriction that D be a function of $\bar{x}$ and s only is imposed. It is assumed that there exist (a) a positive d such that $L_1 + d < \mu < L_2 - d$, (b) a positive C such that $\sigma < C$. From these it follows that there exists a lower bound $G = G(d, C)$ on γ. Let $\chi_{1-\alpha}^2$ be that number for which $P\{\chi^2 < \chi_{1-\alpha}^2\} = 1 - \alpha$, where χ^2 has $N-1$ degrees of freedom, and let

$w=(N-1)^{1/2}s/\chi_{1-\alpha}$. It is shown that if D be defined as follows: (1) if $L_1 \leqq \bar{x} \leqq L_2$, $D=(2\pi)^{-1/2}\int_{(L_1-\bar{x})/w}^{(L_2-\bar{x})/w} \exp\{-y^2/2\}dy$, (2) $D=G$ otherwise, then $|P\{D \leqq \gamma\}-\alpha|$ approaches zero as $N\rightarrow\infty$. Thus D is a large sample lower confidence limit. The extension to upper and two-sided limits presents no difficulty. (Received July 8, 1946.)

Topology

336. R. F. Arens: *Convex topological algebras.* Preliminary report.

A convex topological algebra A is a convex topological linear space in which a multiplication of elements is defined, which is, as is addition and scalar multiplication (for definiteness, take the case of real scalars), continuous simultaneously in both factors. This is a generalization of the concept of normed rings. However, the elements with inverses do not form an open set, nor is inversion continuous when possible. The author proves that if A is a division algebra, and is complete in some metric, then A is finite-dimensional, and hence its structure follows from Frobenius' theorem. This result is fundamental for the representation theory of convex topological algebras. (Received June 7, 1946.)

337. R. F. Arens: *Duality in topological linear spaces.*

Let L be any topological linear space with elements x. Let L^* be the set of continuous linear functionals f defined on L, and use in L^* the topology in which convergence of directed sets means uniform convergence on each compact subset of L (the k-topology). Let this construction be repeated, using L^* instead of L, and giving rise to L^{**}, with elements X. Then for each X there is an $x\in L$ such that $X(f)=f(x)$ for each $f\in L^*$. This natural mapping from L^{**} back into L is 1-1 and continuous if L is convex; if furthermore L is complete in some invariant metric (in particular, if L is a Banach space) then the natural mapping is bicontinuous. (Received July 10, 1946.)

338. R. H. Bing: *Skew sets.*

No plane set G contains a collection of five mutually separated sets such that the closure of the sum of any pair of these five sets is the closure of a connected subset of G which is open in G. (Received July 10, 1946.)

339. G. D. Birkhoff and D. C. Lewis: *Chromatic polynomials.*

The number of ways a map P_{n+3} of $n+3$ regions can be colored with λ colors is given by a polynomial $P_{n+3}(\lambda)$ of degree $n+3$. Certain new properties of these chromatic polynomials are established. For instance, if P_{n+3} is regular and if $(-1)^h a_h$ is the coefficient of $(\lambda-2)^{n-h}$ in the expansion of $P_{n+3}(\lambda)/\lambda(\lambda-1)(\lambda-2)$ in powers of $\lambda-2$, it is shown that binomial coefficient $C_h^n \leqq a_h \leqq C_h^{n+h-1}$. Similar results are obtained for expansions in powers of $\lambda-5$. Moreover, extensive numerical calculations indicate that both $P_{n+3}(\lambda)/\lambda(\lambda-1)(\lambda-2)-(\lambda-3)^n$ and $(\lambda-2)^n-P_{n+3}(\lambda)/\lambda(\lambda-1)(\lambda-2)$ are positively completely monotonic for $\lambda \geqq 4$. This conjecture is a very strong form of the usual four-color proposition that $P_n(4)>0$. In connection with reducibility, reduction formulas, and the analysis of rings, the theory of Kempe chains, which has been applied qualitatively with considerable success to the case $\lambda=4$, is generalized so as to yield quantitative results on chromatic polynomials for all values of λ. Typical results on reducible configurations, previously obtained only by use of Kempe chains, are also obtained inductively. The present paper therefore to some extent attempts to