CYLINDERS IN A CONE

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1. **The two problems.** Let B_0 be the $(k-1)$-volume[1] of a figure that lies in a $(k-1)$-dimensional hyperplane of the k-dimensional euclidean space R_k. Throughout this paper k will be a fixed integer greater than unity. Let Q be any point in R_k, not a point of the hyperplane containing B_0, and let h be the length of the altitude drawn from Q to the hyperplane containing B_0. If Q is joined to each point of B_0 by

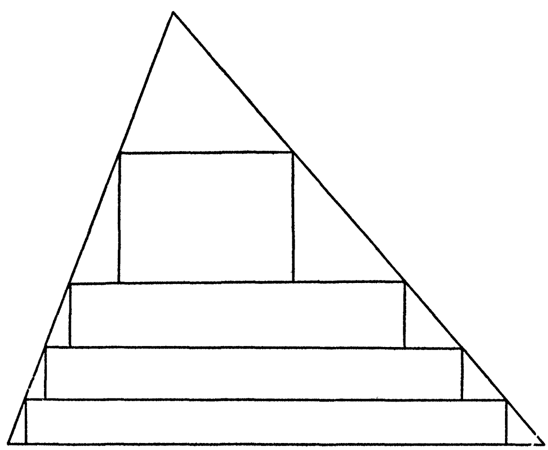

FIG. 1.

a line, the resulting figure is a k-dimensional cone whose k-volume V is given by $V=B_0h/k$. If P_0 is the foot of the above altitude, choose n points $P_1, P_2, \cdots, P_n$ on P_0Q in the natural order $P_0, P_1, P_2, \cdots, P_n, Q$. Through P_i $(i=1, 2, \cdots, n)$ draw a hyperplane parallel to B_0 cutting the cone in a $(k-1)$-dimensional figure B_i which is similar to B_0. Let V_{in} be the k-volume of the right cylinder one of whose bases is B_i while the opposite base lies in the hyperplane containing B_{i-1} $(i=1, 2, \cdots, n)$. Let $X_n=V_{1n}+V_{2n}+\cdots+V_{nn}$ and let $x_{1n}, x_{2n}, \cdots, x_{nn}$ be the altitudes of the cylinders $V_{1n}, V_{2n}, \cdots, V_{nn}$, respectively. (Here x_{in} is the length of $P_{i-1}P_i$.)

Figures 1 and 2 illustrate the cases $k=2$, $n=4$ and $k=3$, $n=3$, respectively.

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[1] By "m-volume" we mean the m-dimensional content; thus 1-volume=length, 2-volume=area, and so on.

Our first problem is to obtain the maximum of X_n for fixed n when no restrictions other than the above are placed upon the V_{in}. Our second problem is to obtain the maximum of X_n for fixed n under the added condition that $V_{1n}=V_{2n}=\cdots=V_{nn}$. We shall refer to these problems hereafter as the *first* and *second problems*, respectively.[2] In order to avoid ambiguity in notation we shall denote those values of the variables $X_n, x_{1n}, x_{2n}, \cdots, x_{nn}$ which correspond to the solution of the first problem by $S_n, s_{1n}, s_{2n}, \cdots, s_{nn}$, respectively, and those values which correspond to the solution of the second problem by $T_n, t_{1n}, t_{2n}, \cdots, t_{nn}$, respectively.

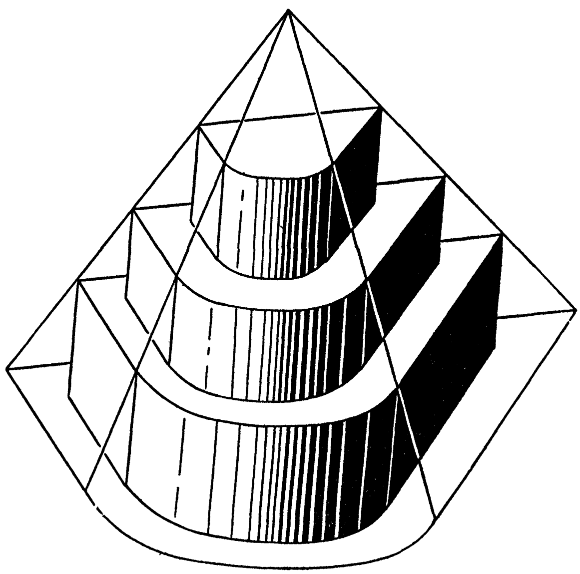

FIG. 2.

In the first problem we show that

$$S_n = y_n^{k-1}V, \tag{1}$$

$$\begin{aligned} s_{1n} &= (1-y_n)h, \\ s_{in} &= y_n y_{n-1} \cdots y_{n-i+2}(1-y_{n-i+1})h, \qquad i = 2, 3, \cdots, n; \end{aligned} \tag{2}$$

where the numbers y_n are defined by the recursion formula

$$y_0 = 0, \qquad y_n = (k-1)/(k-y_{n-1}^{k-1}), \qquad n = 1, 2, \cdots. \tag{3}$$

[2] The second problem for the case $k=2$, $n=2$, but for a quarter-circle rather than a triangle, was treated in the following paper: B. M. Stewart, *Two rectangles in a quarter-circle*, Amer. Math. Monthly vol. 52 (1945) pp. 92–94.

In the second problem we show that

$$(4) \qquad T_n = knu_nV,$$

$$(5)\ t_{in} = u_n[(1+u_{n-1})(1+u_{n-2})\cdots(1+u_{n-i})]^{k-1}h,\ i = 1, 2, \cdots, n;$$

where the numbers u_n are defined by the recursion formula

$$(6) \qquad u_0 = 1/(k-1), \qquad u_n = u_{n-1}/(1+u_{n-1})^k, \qquad n = 1, 2, \cdots.$$

2. **Proof.** The following formulas which will be needed in the proof follow easily from the proportion $B_1/(h-x_{1n})^{k-1} = B_0/h^{k-1}$:

$$(7) \qquad V_{1n} = B_0x_{1n}(h - x_{1n})^{k-1}/h^{k-1},$$

$$(8) \qquad V' = B_0(h - x_{1n})^k/kh^{k-1},$$

where V' is the k-volume of the cone whose base is B_1 and whose altitude P_1Q has the length $h-x_{1n}$.

The proofs of the results for both problems are by induction with respect to n. The two problems are identical when $n=1$, and by elementary calculus it follows readily from (7) together with $X_1 = V_{11}$ that $s_{11} = t_{11} = h/k$ and that $S_1 = T_1 = [(k-1)/k]^{k-1}V$. These results agree with (1), (2), (4), and (5) when $n=1$.

In the first problem we assume that formulas (1) and (2) are correct for $n-1$ where $n \geqq 2$. Let x_{1n} be chosen arbitrarily $(0 < x_{1n} < h)$ and, depending on the choice of x_{1n}, let $x_{2n}, x_{3n}, \cdots, x_{nn}$ be chosen so as to maximize the combined k-volume of the remaining $n-1$ cylinders, namely:

$$X_{n-1}' = V_{2n} + V_{3n} + \cdots + V_{nn}.$$

By the induction hypothesis (see (1)) the maximum of X_{n-1}' is given by $S_{n-1}' = y_{n-1}^{k-1}V'$. We thus obtain, for each choice of x_{1n}, a maximum value of X_n, namely, $X_n = V_{1n} + S_{n-1}'$. Considering this X_n as a function of x_{1n}, namely (see (7) and (8)),

$$(9) \quad X_n = B_0x_{1n}(h - x_{1n})^{k-1}/h^{k-1} + y_{n-1}^{k-1}B_0(h - x_{1n})^k/kh^{k-1},$$

we see that X_n reaches its maximum value S_n when x_{1n} has the value

$$s_{1n} = \frac{1 - y_{n-1}^{k-1}}{k - y_{n-1}^{k-1}}\, h = \left(1 - \frac{k-1}{k - y_{n-1}^{k-1}}\right) h,$$

or by (3) we may write

$$(10) \qquad s_{1n} = (1 - y_n)h,$$

in agreement with (2). Substituting this value of s_{1n} for x_{1n} in (9) and using (3) we obtain (1).

It remains to show (2) for $i = 2, 3, \cdots, n$. Note that S_{n-1}' represents the solution of the first problem for V'. Hence if the altitudes of these $n-1$ cylinders are denoted by $s_{1,n-1}'$, $s_{2,n-1}'$, $\cdots$, $s_{n-1,n-1}'$, we have $s_{in}=s_{i-1,n-1}'$ for $i=2, 3, \cdots, n$ and hence by the induction hypothesis (see (2)) $s_{in}=y_{n-1}y_{n-2}\cdots y_{n-i+2}(1-y_{n-i+1})(h-s_{1n})$. By (10) this establishes (2) for $i=2, 3, \cdots, n$. This completes the proof of (1) and (2).

In the second problem, the case $n=1$ having been disposed of above, we now assume that (4) and (5) are correct for $n-1$, where $n\geqq 2$. We shall choose x_{1n} arbitrarily but such that it is possible to inscribe $n-1$ cylinders of k-volume equal to V_{1n} in V'. By the induction hypothesis (see (4)) this is possible if and only if

$$(n-1)V_{1n} \leqq k(n-1)u_{n-1}V'. \tag{11}$$

By virtue of (7) and (8) and the fact that $0<x_{1n}<h$, the inequality (11) is equivalent to

$$0 < x_{1n} \leqq u_{n-1}h/(1+u_{n-1}). \tag{12}$$

Our problem is therefore to maximize the quantity

$$X_n = nV_{1n} = nB_0x_{1n}(h-x_{1n})^{k-1}/h^{k-1} \tag{13}$$

within the interval (12). But the function X_n increases over the interval $0<x_{1n}\leqq h/k$ and the right-hand end point of the interval (12) is less than h/k. (This follows easily from (6): the u_n form a decreasing sequence of positive numbers, hence $u_{n-1}/(1+u_{n-1})<u_{n-1}\leqq u_1 =(k-1)^{k-1}/k^k<1/k$, $n=2, 3, \cdots$.) Therefore the function X_n assumes its maximum in the interval (12) at the right-hand end point, so that we obtain $t_{1n}=u_{n-1}h/(1+u_{n-1})$ and by (6) we may write

$$t_{1n} = u_n(1+u_{n-1})^{k-1}h, \tag{14}$$

$$h - t_{1n} = h/(1+u_{n-1}). \tag{15}$$

Substituting the values of t_{1n} and $h-t_{1n}$ from (14) and (15) for x_{1n} and $h-x_{1n}$ in (13) we obtain (4).

When x_{1n} assumes the value t_{1n} given in (14), the inequality (11) becomes an equation. Consequently, the $n-1$ cylinders V_{2n}, V_{3n}, $\cdots$, V_{nn} must be the solution of the second problem for V'. Hence if the altitudes of these cylinders are denoted by $t_{1,n-1}'$, $t_{2,n-1}'$, $\cdots$, $t_{n-1,n-1}'$, we have $t_{in}=t_{i-1,n-1}'$ for $i=2, 3, \cdots, n$ and hence, by the induction hypothesis (see (5)),

$$t_{in} = u_{n-1}[(1+u_{n-2})(1+u_{n-3})\cdots(1+u_{n-i})]^{k-1}(h-t_{1n}).$$

By (15) and (6) this establishes (5) for $i=2, 3, \cdots, n$. Since (14) agrees with (5) for $i=1$, this completes the proof.

3. **Asymptotic formulas for S_n and T_n.** The problem arises whether the quantities S_n and T_n, given by (1) and (4), respectively, can be expressed directly in terms of n. This seems possible only for the S_n in the case $k=2$, that is, for the problem of maximizing the combined area of n rectangles inscribed in a triangle. Indeed in this case (3) becomes $y_n=1/(2-y_{n-1})$, which together with $y_0=0$ yields easily by induction that $y_n=n/(n+1)$. Hence from (2) and (1) we obtain $s_{in}=h/(n+1)$ for $i=1, 2, \cdots, n$ and $S_n=nV/(n+1)$ or

$$S_n/V = 1 - 1/(n+1).$$

Thus the problem arises to give at least an asymptotic formula for S_n/V when $k\geqq 3$, as well as an asymptotic formula for T_n/V when $k\geqq 2$.

We begin by establishing an asymptotic formula for the y_n (in the case $k\geqq 3$). We put

$$z_n = 1 - y_n \tag{16}$$

and obtain from (3)

$$z_0 = 1, \qquad 1/z_n = F(z_{n-1}), \qquad n = 1, 2, \cdots, \tag{17}$$

where

$$F(z) = \frac{k-(1-z)^{k-1}}{1-(1-z)^{k-1}}. \tag{18}$$

It is easily established from (3) by mathematical induction that $(k-1)/k\leqq y_n<1$ so that by (16)

$$0 < z_n \leqq 1/k, \qquad n = 1, 2, \cdots. \tag{19}$$

We shall need the two following facts about $F(z)$, defined in (18). In the first place,

$$F(z) = 1 + \frac{k-1}{C_{k-1,1}z - C_{k-1,2}z^2 + \cdots + (-1)^k z^{k-1}}.$$

The terms in the denominator on the right are decreasing in absolute value when $0<z<2/(k-2)$, hence for such values we have $F(z)>1+1/z$, so that in particular by (19), since $1/k<2/(k-2)$,

$$F(z_j) > 1 + 1/z_j, \qquad j = 1, 2, \cdots. \tag{20}$$

Secondly, we conclude from (18) that $F(z)$ is a regular function except for poles at the points $z=0$ and $z=1-\exp[2\pi i m/(k-1)]$ with $m=1, 2, \cdots, k-2$, so that $F(z)$ admits of a Laurent expansion in the region $0<|z|<|1-\exp[2\pi i/(k-1)]|=2\sin[\pi/(k-1)]$. This Laurent series is easily seen to be

$$(21) \qquad F(z) = 1/z + k/2 + G(z), \qquad G(z) = \sum_{m=1}^{\infty} a_m z^m,$$

where the power series $G(z)$ converges for $|z|<2\sin[\pi/(k-1)]$. Since $1/k<4/(k-1)\leqq 2\sin[\pi/(k-1)]$ for $k\geqq 3$, we conclude that $G(z)/z$ is bounded for $|z|\leqq 1/k$. The latter fact together with (19) yields

$$(22) \qquad G(z_n) = O(z_n).^{3}$$

From (17) and (20) we have $1/z_j>1+1/z_{j-1}$, $j=2, 3, \cdots$. By adding these inequalities from $j=2$ to $j=n$, we obtain $1/z_n>n-1+1/z_1$; hence

$$(23) \qquad z_n = O(1/n).$$

From (17) and (21) we conclude that $1/z_j=1/z_{j-1}+k/2+G(z_{j-1})$, $j=2, 3, \cdots$. By adding these equations from $j=2$ to $j=n$, we obtain

$$(24) \qquad 1/z_n = 1/z_1 + k(n-1)/2 + \sum_{i=1}^{n-1} G(z_i).$$

Applying (22) and (23) to the $G(z_i)$ in (24), we thus obtain

$$1/z_n = kn/2 + O(\log n) = (kn/2)[1 + O(n^{-1}\log n)],$$
$$z_n = (2/kn)[1 + O(n^{-1}\log n)] = 2/kn + O(n^{-2}\log n).$$

Therefore by substituting this result in (1), we obtain the following asymptotic formula, valid for $k\geqq 3$:

$$S_n/V = y_n{}^{k-1} = (1 - z_n)^{k-1} = [1 - 2/kn + O(n^{-2}\log n)]^{k-1},$$
$$S_n/V = 1 - 2(k-1)/kn + O(n^{-2}\log n).$$

To establish an asymptotic formula for T_n/V when $k\geqq 2$, we begin with an asymptotic formula for the u_n. We write (6) in the form $1/u_j=(1+u_{j-1})^k/u_{j-1}$ or

$$(25) \qquad 1/u_j = 1/u_{j-1} + k + C_{k,2}u_{j-1} + \sum_{m=3}^{k} C_{k,m}u_{j-1}{}^{m-1},$$

where the $\sum$ in (25) is to mean zero when $k=2$. By adding these equations from $j=2$ to $j=n$, we obtain

[3] The notation $f(n)=O(\phi(n))$ is used here to mean that $|f(n)|<A\phi(n)$ for sufficiently large n, where A is independent of n but may depend on k. In particular we shall use the fact that $\sum_{i=1}^{n-1}(1/i)^m$ is of the form $\log n+O(1)$ when $m=1$ and of the form $O(1)$ when $m>1$. Also if $\phi(n)\to 0$ the reciprocal value of $1+O(\phi(n))$ is again of the form $1+O(\phi(n))$.

$$(26)\quad 1/u_n = 1/u_1 + k(n-1) + C_{k,2}\sum_{i=1}^{n-1} u_i + \sum_{m=3}^{k} C_{k,m}\sum_{i=1}^{n-1} u_i{}^{m-1}$$

for $n=2, 3, \cdots$. Hence since $u_i>0$ for $i=1, 2, \cdots$, we have $1/u_n>k(n-1)$ and

$$(27)\qquad u_n = O(1/n).$$

Applying (27) to the u_i on the right of (26), we have (see footnote 3)

$$1/u_n = kn + O(\log n) = kn[1 + O(n^{-1}\log n)],$$

$$(28)\qquad u_n = (1/kn)[1 + O(n^{-1}\log n)] = 1/kn + O(n^{-2}\log n).$$

Using (28) in (26) we have, since $\sum_{i=1}^{n-1} i^{-2}\log i = O(1)$,

$$1/u_n = kn + 2^{-1}(k-1)\log n + O(1)$$

$$= kn\left[1 + \frac{k-1}{2k}\,\frac{\log n}{n} + O\left(\frac{1}{n}\right)\right],$$

$$u_n = \frac{1}{kn}\left[1 - \frac{k-1}{2k}\,\frac{\log n}{n} + O\left(\frac{1}{n}\right)\right].$$

Finally, substituting this result in (4), we obtain the following asymptotic formula:

$$T_n/V = 1 - \frac{k-1}{2k}\,\frac{\log n}{n} + O\left(\frac{1}{n}\right).$$

4. Table. In conclusion we append a brief table which indicates how involved the numbers s_{in}, S_n, t_{in}, T_n become, even for small values of k and n.

First problem			Second problem		
	$k=2$	$k=3$		$k=2$	$k=3$
s_{11}/h	1/2	1/3	t_{11}/h	1/2	1/3
S_1/V	1/2	4/9	T_1/V	1/2	4/9
s_{12}/h	1/3	5/23	t_{12}/h	1/5	4/31
s_{22}/h	1/3	6/23	t_{22}/h	2/5	9/31
S_2/V	2/3	324/529	T_2/V	16/25	17496/29791

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