QUADRICS ASSOCIATED WITH A CURVE ON A SURFACE

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1. Introduction. Many of the important contributions to projective differential geometry of non-ruled surfaces are concerned with systems of quadrics associated with a point and a curve on the surface. Many of these quadrics belong to a certain family, a characterization of which is the main purpose of this paper.

Let the homogeneous projective coordinates (x^1, x^2, x^3, x^4) of a general point x on a non-ruled surface S be given as functions of the asymptotic parameters u, v, and let these functions be so normalized that they satisfy the Fubini canonical system of differential equations,

$$x_{uu} = \theta_u x_u + \beta x_v + p x,$$
$$x_{vv} = \gamma x_u + \theta_v x_v + q x, \qquad \theta = \log (\beta\gamma),$$

wherein the coefficients satisfy certain integrability conditions [7].[1] The abbreviations

$$\phi = \partial \log (\beta\gamma^2)/\partial u, \qquad \psi = \partial \log (\beta^2\gamma)/\partial v$$

will be found useful.

Let C_λ, a curve on S through x, be considered as imbedded in a one-parameter family of curves defined by the differential equation

$$dv - \lambda du = 0.$$

Since the homogeneous coordinates of any point X may be written in the form

$$X = x_1 x + x_2 x_u + x_3 x_v + x_4 x_{uv},$$

the coordinates of X referred to the tetrahedron x, x_u, x_v, x_{uv} may be taken as (x_1, x_2, x_3, x_4).

It is remarkable that many of the equations of quadrics associated with S and C_λ at x are of the form

$$x_2 x_3 + T x_4 = 0 \tag{1}$$

wherein

$$T = - x_1 + k_2 x_2 + k_3 x_3 + k_4 x_4, \tag{2}$$

and

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[1] Numbers in brackets refer to the references cited at the end of the paper.

(3) $$k_2 = l_2\beta/\lambda + m_2\gamma\lambda^2, \qquad k_3 = l_3\beta/\lambda^2 + m_3\gamma\lambda,$$

l_2, m_2, l_3, m_3 being constants and k_4 a parameter. In particular the quadrics of Darboux, of Moutard, of Davis [3], all of the quadrics derived by Wu [9], the conjugal quadrics [5], the asymptotic osculating quadrics, and the quadrics Hsiung [6] has associated with C_λ at x all belong to the system (1). We shall denote this family by $Q(l_2, m_2, l_3, m_3)$.

2. **A characterization of the family.** Let a line l be determined by the points $\rho\sigma$ with coordinates given by the expressions $\rho = x_u - bx$, $\sigma = x_v - ax$. The R_λ-associate of l, as defined by Bell [1], joins the points whose coordinates are

(4) $$\rho_\lambda = \rho + \beta x/\lambda, \qquad \sigma_\lambda = \sigma + \gamma\lambda x.$$

Bell [1] has called the one-parameter family of curves defined by

$$dv - \mu du = 0, \qquad \mu = -\beta/(\gamma\lambda^2)$$

the R_λ-derived curves, and has characterized them in terms of the R_λ-associate of l. The R_μ-associate of l joins the points

(5) $$\rho_\mu = \rho - \gamma\lambda^2 x, \qquad \sigma_\mu = \sigma - \beta x/\lambda^2.$$

From (4) and (5) we easily prove the following theorems.

The R_λ-associate of l coincides with the R_μ-associate of l if and only if C_λ is a curve of Darboux.

The R_λ-associate of l, l, and the R_μ-associate of l intersect the asymptotic tangents in points which with x are harmonic if and only if C_λ is a curve of Segre.

Now define points R, S, R_λ, S_λ by the cross ratio equations

(6) $$\begin{aligned} (x, \rho_\lambda, \rho_\mu, R) &= k, \qquad & (x, R, \rho, R_\lambda) &= K, \\ (x, \sigma_\lambda, \sigma_\mu, S) &= l, \qquad & (x, S, \sigma, S_\lambda) &= L, \end{aligned}$$

k, l, K, L being constants. One finds readily that the coordinates of R_λ, S_λ are given by the expressions

(7) $$R_\lambda = \rho + k_2 x, \qquad S_\lambda = \sigma + k_3 x$$

wherein k_2, k_3 are given by (3) and

(8) $$\begin{aligned} l_2 &= kK, \qquad & m_2 &= K(k-1), \\ l_3 &= L(l-1), \qquad & m_3 &= lL. \end{aligned}$$

We shall call the line joining R_λ, S_λ the $S_\lambda(l_2, m_2, l_3, m_3)$-associate of l.

The reciprocal l' of the line l joins x to the point with coordinates $x_{uv}-ax_u-bx_v$. It is easy to show that *the most general quadric having second order contact with S at x and with respect to which the polar line of l' is the $S_\lambda(l_2, m_2, l_3, m_3)$-associate of l is the quadric $Q(l_2, m_2, l_3, m_3)$.*

3. **The general transformation of Čech.** The coordinates of any point R on the tangent to C_λ at x may be written in the form

$$R = x_u + \lambda x_v + tx.$$

The polar plane of R with respect to $Q(l_2, m_2, l_3, m_3)$ has coordinates (u_1, u_2, u_3, u_4) defined by the formulas

$$\begin{aligned} u_1 = 0, \qquad u_2 = \lambda^2, \qquad u_3 = \lambda, \\ u_4 = -\lambda t + (l_2 + l_3)\beta + (m_2 + m_3)\gamma\lambda^3. \end{aligned} \tag{9}$$

If the local coordinates of R be written as $(x_1, x_2, x_3, 0)$, equations (9) assume the form

$$\begin{aligned} u_1 = 0, \qquad u_2 = x_2x_3^2, \qquad u_3 = x_2^2x_3, \\ u_4 = -x_1x_2x_3 + (l_2 + l_3)\beta x_2^3 + (m_2 + m_3)\gamma x_3^3 \end{aligned} \tag{10}$$

of the most general transformation of Čech [2].

Bell [1] has given a geometric characterization of this general transformation. Lane [7] has characterized this transformation for the special case $l_2+l_3=m_2+m_3$.

The quadric $Q_{-\lambda}(l_2, m_2, l_3, m_3)$ induces the transformation

$$\begin{aligned} u_1 = 0, \qquad u_2 = x_2x_3^2, \qquad u_3 = x_2^2x_3, \\ u_4 = -x_1x_2x_3 + (l_2 - l_3)\beta x_2^3 + (m_2 - m_3)\gamma x_3^3. \end{aligned} \tag{11}$$

The particular quadrics referred to above induce several interesting special transformations (10) and (11).

4. **The quadrics** $Q_u(l_2, m_2, l_3, m_3)$, $Q_v(l_2, m_2, l_3, m_3)$. We may characterize some of the quadrics in the family (1) in the following manner. The $S_\lambda(l_2, m_2, l_3, m_3)$-associate of the reciprocal of the projective normal joins the points defined by $R_\lambda=x_u+k_2x$, $S_\lambda=x_v+k_3x$, k_2, k_3 being defined by (3). The tangent to the locus of R_λ as x moves along C_λ and the point S_λ determine a plane π. The plane π intersects the projective normal in a point P. As x moves along C_λ the plane $x_3=0$ envelops a developable surface generated by a line which intersects the projective normal in a point Q. The plane determined by R_λ, S_λ and the harmonic conjugate of x with respect to P and Q has the equation $T=0$, T being defined by (2) and wherein

$$
\begin{aligned}
k_4 = \frac{1}{2}\Big\{ &- \frac{\beta}{\lambda^3}\Big[l_2\lambda' + (l_2^2 + l_2 l_3 + l_3)\beta \\
&+ \Big[(1 - l_2)\frac{\beta_v}{\beta} + \theta_v\Big]\lambda^2 + l_2\Big(\theta_u - \frac{\beta_u}{\beta}\Big)\lambda\Big] \\
&+ m_2\gamma\Big[2\lambda' - (m_2 + m_3)\gamma\lambda^3 + \frac{\gamma_v}{\gamma}\lambda^2 + \Big(\frac{\gamma_u}{\gamma} - \theta_u\Big)\lambda\Big] \\
&- \theta_{uv} - [(2l_2 + l_3)m_2 + m_3(1 + l_2) + 1]\beta\gamma\Big\}.
\end{aligned} \tag{12}
$$

The quadric having second order contact with S at x and passing through the lines $x_2=0$, $T=0$; $x_3=0$, $T=0$ has the equation (1), k_4 being given by (12). We shall denote this quadric by $Q_u(l_2, m_2, l_3, m_3)$.

In an analogous manner we may define a quadric $Q_v(l_2, m_2, l_3, m_3)$ with the equation (1) with k_4 given by the formula

$$
\begin{aligned}
k_4 = \frac{1}{2}\Big\{ &- \gamma\Big[- m_3\lambda' + (m_3^2 + m_2 m_3 + m_2)\gamma\lambda^3 \\
&+ \Big[(1 - m_3)\frac{\gamma_u}{\gamma} + \theta_u\Big]\lambda + m_3\Big(\theta_v - \frac{\gamma_v}{\gamma}\Big)\lambda^2\Big] \\
&+ \frac{l_3\beta}{\lambda^3}\Big[- 2\lambda' - (l_2 + l_3)\beta + \frac{\beta_u}{\beta}\lambda + \Big(\frac{\beta_v}{\beta} - \theta_v\Big)\lambda^2\Big] \\
&- \theta_{uv} - [(2m_3 + m_2)l_3 + l_2(1 + m_3) + 1]\beta\gamma\Big\}.
\end{aligned} \tag{13}
$$

5. **Applications.** It is easily verified that *the special quadrics* $Q_u(-1, 0, 1, 0)$, $Q_v(0, 1, 0, -1)$ *are the asymptotic osculating quadrics of the curve* C_λ *at* x. These quadrics may therefore be considered as generalizations of the asymptotic osculating quadrics. *The quadric* $Q_u(0, 0, 0, 0)$ *for a curve* C_λ *tangent to the asymptotic* $u=$const. (*or* $Q_v(0, 0, 0, 0)$ *for a curve* C_λ *tangent to* $v=$const.) *is the quadric of Lie.*

Let us call the quadrics $Q_u(0, 1, 0, -1)$, $Q_v(-1, 0, 1, 0)$ *the anti-asymptotic osculating quadrics*. They are given by (1) with the respective values of k_4:

$$
\begin{aligned}
k_4 = \frac{1}{2}\Big\{ &\gamma\Big[2\lambda' + \frac{\gamma_v}{\gamma}\lambda^2 + \Big(\frac{\gamma_u}{\gamma} - \theta_u\Big)\lambda\Big] \\
&- \frac{\beta}{\lambda}\Big(\frac{\beta_v}{\beta} + \theta_v\Big) - \theta_{uv}\Big\},
\end{aligned}
$$

$$k_4 = \frac{1}{2}\left\{\frac{\beta}{\lambda^3}\left[-2\lambda' + \frac{\beta_u}{\beta}\lambda + \left(\frac{\beta_v}{\beta} - \theta_v\right)\lambda^2\right]\right.$$
$$\left. - \gamma\lambda\left(\frac{\gamma_u}{\gamma} + \theta_u\right) - \theta_{uv}\right\}.$$

It is easy to verify the following theorems.

The anti-asymptotic osculating quadrics (assumed distinct) intersect in the asymptotic tangents and in a conic whose plane passes through the projective normal if and only if C_λ is a pangeodesic. These quadrics coincide if and only if $\beta+\gamma\lambda^3=0$, and $\phi+\lambda\psi=0$; that is, the curve C_λ must be tangent to a curve of Darboux, and that tangent must be the second canonical tangent of S at x.

The conjugal quadrics $Q_u(0, k, k, 0)$, $Q_v(0, k, k, 0)$ have equations given by (1) with the respective values of k_4:

$$k_4 = \frac{1}{2}\left\{k\gamma\left[2\lambda' - k\gamma\lambda^3 + \frac{\gamma_v}{\gamma}\lambda^2 + \left(\frac{\gamma_u}{\gamma} - \theta_u\right)\lambda\right]\right.$$
$$\left. - \frac{\beta}{\lambda^3}\left[k\beta + \left(\frac{\beta_v}{\beta} + \theta_v\right)\lambda^2\right] - \theta_{uv} - (1 + k^2)\beta\gamma\right\},$$

$$k_4 = \frac{1}{2}\left\{- k\frac{\beta}{\lambda^3}\left[2\lambda' + k\beta - \frac{\beta_u}{\beta}\lambda - \left(\frac{\beta_v}{\beta} - \theta_v\right)\lambda^2\right]\right.$$
$$\left. - \gamma\left[k\gamma\lambda^3 + \left(\frac{\gamma_u}{\gamma} + \theta_u\right)\lambda\right] - \theta_{uv} - (1 + k^2)\beta\gamma\right\}.$$

These quadrics coincide if and only if

$$\begin{aligned} 2k(\beta + \gamma\lambda^3)\lambda' = {} & k(1 - k)\beta^2 + k\beta_u\lambda \\ & + \beta\lambda^2\left[(1 + k)\frac{\beta_v}{\beta} + (1 - k)\theta_v\right] \\ & - \gamma\lambda^4\left[(1 + k)\frac{\gamma_u}{\gamma} + (1 - k)\theta_u\right] \\ & - k\gamma_u\lambda^5 - k(1 - k)\gamma^2\lambda^6. \end{aligned} \tag{14}$$

The curves represented by (14) *are hypergeodesics if and only if $\phi=\psi=0$, that is, S is a coincidence surface.* In that case the hypergeodesics represented by (13) are given by

$$\lambda' = \frac{1}{2}(1 - k)\beta + \frac{1}{2}\frac{\beta_u}{\beta}\lambda - \frac{1}{2}\frac{\gamma_v}{\gamma}\lambda^2 - \frac{1}{2}(1 - k)\gamma\lambda^3.$$

The cusp-axis of these hypergeodesics is the projective normal. From (14) we see that *the quadrics* $Q_u(0, 1, 1, 0)$, $Q_v(0, 1, 1, 0)$ *of Davis coincide if and only if* C_λ *is a pan-geodesic. And the curves* C_λ *defined by* (14) *are pan-geodesics if and only if the quadrics are quadrics of Davis.*

As the point x moves along C_λ the asymptotic tangents generate ruled surfaces R_u, R_v. Hsiung [6] has shown the existence of a pair of quadrics associated with R_u and R_v. He cuts these surfaces by a plane through the points $P_u = x_u + Ax$, $P_v = x_v + Bx$. The locus of the conic having ordinary contact at P_u and second order contact at P_v with these sections is a quadric $Q(-1, 0, 0, -1)$ whose equation is (1) with k_4, given by the formula

$$k_4 = \frac{1}{2}\left\{\frac{\beta}{\lambda^3}\left[\lambda' - \lambda(\theta_u - \theta_v\lambda - \phi + 2\psi\lambda)\right] + \frac{\beta^2}{\lambda^3} + 2A\lambda(\beta + \gamma\lambda^3) + \beta\gamma - \theta_{uv}\right\}.$$

This quadric of Hsiung coincides with $Q_u(-1, 0, 0, -1)$ *if and only if* P_u *is the point whose coordinates are given by the expression*

$$P_u = x_u - \beta x/\lambda. \tag{15}$$

Interchanging the roles of the asymptotic curves and ruled surfaces, a second point P_v with coordinates

$$P_v = x_v - \gamma\lambda x \tag{16}$$

is characterized.

The R_λ*-associate of the reciprocal of the projective normal, that reciprocal, the line* P_uP_v *defined by* (15) *and* (16), *and the tangent to the* R_λ*-derived curve through* x *are concurrent, and are moreover harmonic. The* R_λ*-associate of the line* P_uP_v *is the projective normal.*

It is known [7] that the asymptotic osculating quadric $Q_v(0, 1, 0, -1)$ reduces to the quadric of Wilczynski if C_λ is tangent to the asymptotic curve $v = \text{const.}$, and has an inflexion at x, and to the quadric of Fubini if C_λ is tangent to $v = \text{const.}$, and has the tangent plane to S at x as stationary osculating plane. These quadrics are respectively the special quadrics $h = 1$, $h = 1/3$ of the pencil

$$x_2x_3 + x_4\{-x_1 - (1/2)[\theta_{uv} + (1 - h)\beta\gamma]x_4\} = 0. \tag{17}$$

Lane [8] has characterized the invariant parameter h of the pencil (17) in terms of a cross ratio whose elements involve the quadric of Lie ($h = 0$) and the quadric of Wilczynski ($h = 1$). The definition (8) of m_3 enables us to describe the invariant parameter h without recourse to any special quadric of (17).

The quadric $Q_v(0, m_2, 0, m_3)$ for a curve C_λ tangent to $v=$const. and having an inflexion at x ($\lambda=0, \lambda'+\beta=0$) has the equation (17) with $h=-m_3$. The definitions (8) imply that $l=1$, $k=0$, $m_3=L$, $m_2=K$. From (6) we find that $R=\rho_\mu$, $S=\sigma_\lambda$. Then the points R_λ, S_λ determining the $S_\lambda(0, m_2, 0, m_3)$-associate of l are found from the cross ratio equations

$$(x, \rho_\mu, \rho, R_\lambda) = K, \qquad (x, \sigma_\lambda, \sigma, S_\lambda) = L = -h.$$

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