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AUTOMORPHISMS OF FIELDS OF FORMAL POWER SERIES

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We propose to discuss in this note on power series fields in one variable the special automorphisms which do not alter the fields of coefficients. It will be proved that the pseudo-ramification groups introduced by MacLane are universal ramification groups, in the sense that a special ramification group must always be a subgroup of a well determined pseudo-ramification group. Finally we interpret the automorphism group of the field as an automorphism group of an infinite Lie ring.

Let Ω be an arbitrary field of characteristic χ. In the sequel we shall consider the field F of all formal power series $a=\sum_{j>-\infty}^{\infty}\omega_j t^j$ where the ω_j are in Ω and t is a transcendental element over Ω.[1] The field F is complete with respect to the rank one valuation V defined by $Va=m$ where m is the smallest subscript j for which $\omega_j\neq 0$. Let $\mathfrak{O}$ be the valuation ring of all holomorphic series and $\mathfrak{P}=(t)$ the principal prime ideal of $\mathfrak{O}$.

Suppose that S is an automorphism of F. We show that $\mathfrak{O}^S$ is also a valuation ring of F. For the proof[2] let a, b be any two nonzero elements of F. We must show that at least one of the quotients a/b, b/a lies in $\mathfrak{O}^S$. By assumption on S there exist unique elements c, d with $c^S=a$, $d^S=b$. Now observe that at least one of the quotients c/d or d/c lies in $\mathfrak{O}$ for $\mathfrak{O}$ is a valuation ring. Therefore at least one of the

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[1] For the basic properties of valuations see [1, 4, 5, 10]. Numbers in brackets refer to the bibliography at the end of the paper.

[2] See [4, p. 165].

elements $(c/d)^S=a/b$ or $(d/c)^S=b/a$ lies in $\mathfrak{O}^S$. Furthermore we observe that the ideal theory of $\mathfrak{O}$ is carried over isomorphically to $\mathfrak{O}^S$. Thus $\mathfrak{O}^S$ defines a rank one valuation V^S on F.

DEFINITION 1.[3] *An automorphism S of F is called analytic if $Va<Vb$ is equivalent to $V^Sa^S\leqq V^Sb^S$ for each pair of elements a, b in F.*

LEMMA 1. *Each automorphism of F is analytic.*

PROOF. We shall prove that F is complete with respect to the valuation V^S. Let $a_n\equiv a_{n-1} \pmod{(\mathfrak{P}^S)^n}$ be an infinite consistent system of congruences with respect to V^S. There exist uniquely determined elements c_n with $c_n^S=a_n$ and we have $c_n\equiv c_{n-1} \pmod{\mathfrak{P}^n}$. Hence there exists an element x in F with $x\equiv c_n \pmod{\mathfrak{P}^n}$, for F is complete with respect to V. Applying S to x we have $x^S\equiv c_n^S\equiv a_n \pmod{(\mathfrak{P}^S)^n}$. Hence F is complete with respect to V^S. Consequently F would be multiply complete if V and V^S were distinct valuations. Thus it would follow[4] that F is algebraically closed, in contradiction to the construction of F. Hence V and V^S are equivalent valuations, that is, S is analytic.

Suppose that s is an automorphism of Ω. If we observe the rules for the addition and multiplication of elements in F, the correspondence $\sum_{j>-\infty}^{\infty}\omega_jt^j\to\sum_{j>-\infty}^{\infty}\omega_j^st^j=(\sum_{j>-\infty}^{\infty}\omega_jt^j)^S$ defines an automorphism of F. These automorphisms of F determine a subgroup of the automorphism group A of F which is isomorphic with the automorphism group of Ω. A simple computation shows that this subgroup is not normal if and only if it is not the trivial group consisting of the identity. Now let T be an arbitrary automorphism of F. The element T either induces an automorphism on $\Omega\subset F$ or it maps Ω into an isomorphic subfield $\Omega^T\subset F$. *We shall consider only those automorphisms of F for which all the elements of Ω are invariant.* These automorphisms form a subgroup G of A. This group G corresponds to the inertial group considered in the theory of normal algebraic extensions of fields with valuations.

LEMMA 2. *A field F which is complete with respect to a rank one valuation has no immediate extensions.*[5]

PROOF. Suppose that K is an immediate extension of F. Let A be an element of K. We shall show that A must lie in F. By assumption the value of A is the value of an element a in F. Thus A/a is a unit

[3] See [5, footnote on p. 373; 12].
[4] See [12].
[5] See [4, p. 191].

and therefore $A/a \equiv \omega_0 \pmod{\mathfrak{P}_K}$, where $\mathfrak{P}_K$ is the prime ideal of K and ω_0 is in Ω. Next there is at least one element a_1 in F with $A/a-\omega_0 \equiv a_1 \pmod{\mathfrak{P}_K^{n_1}}$, where $n_1 = V(A/a-\omega_0)$. By complete induction there exists a sequence of elements $a\omega_0$, $a_0+aa_1\omega_1, \cdots, a\omega_0 + \cdots + aa_1 \cdots a_\nu\omega_\nu, \cdots$ such that $A \equiv a\omega_0 + \cdots + aa_1 \cdots a_\nu \omega_\nu \pmod{\mathfrak{P}_K^m}$ with $m = n_1 + \cdots + n_\nu \to \infty$ for $\nu \to \infty$. This sequence has a limit b in F and $A-b$ has value ∞ in K. Hence $A = b$ in F.[6]

Suppose that $S \neq 1$ is an automorphism of F with $\omega = \omega^S$ for all ω in Ω. Then $t^S \neq t$, for $t = t^S$ implies $a = a^S$ for every a in F. Hence each $S \neq 1$ determines by $t^S = tu(S)$ an element $u(S)$ of the unit group U of F.

THEOREM 1. *The groups* **G** *and* U *are in one-to-one correspondence as sets.*

PROOF. As seen above each automorphism S determines relative to the prime element t a unit $u(S)$. For the converse we shall show that the mapping $t \to t^* = tu$ determines, for given u in U, an automorphism of F. We associate to an arbitrary element $a = \sum_{j>-\infty}^{\infty} \omega_j t^j$ the quantity $a^* = \sum_{j>-\infty}^{\infty} \omega_j (t^*)^j$. The elements a^* form by definition a subfield F^* of F which is isomorphic to F. By construction the valuation V of F induces a valuation V^* on the complete field F^* with $V^*t^* = Vt^* = Vt$ so that F and F^* have the same residue class field. Therefore F is an immediate extension of F^* and hence $F = F^*$ by Lemma 2. Consequently the mapping $a \to a^*$ is an automorphism of F. We remark that the identity elements of **G** and U correspond to each other.

We next give a definition and a set of formulas which can be used to compute explicitly the coefficients of a^* relative to the prime element t. We define the derivative $D_t a$ of $a = \sum \omega_j t^j$ as $\sum j\omega_j t^{j-1}$ and $D_t^i = (1/i!)(d/dt)^i$. Then[7]

$$\begin{aligned} D_t^i(a+b) &= D_t^i a + D_t^i b, \quad D_t^i t^n = C_{n,i} t^{n-i}, \qquad\qquad i \geqq 0, \\ D_t^i(\omega a) &= \omega D_t^i a, \quad \omega \text{ in } \Omega, \\ D_t^i(ab) &= \sum_i D_t^{i_1} a D_t^{i_2} b; \quad i_1 \geqq 0,\ i_2 \geqq 0,\ i_1 + i_2 = i. \end{aligned}$$

As in the calculus the inversion formula of Lagrange holds, $t = \sum_{j=1}^{\infty} \lambda_j (t^*)^j$ with $\lambda_j = [D_t^j \{ D_t t^* \cdot (t/t^*)^j \}]_{t=0}$, $\lambda_1 \neq 0$. Hence t lies in F^* and thus $F^* = F$. Using the Taylor developments of the elements $a^* = \sum \omega_j (t^*)^j = \sum D_t^j \{a^*\}_{t=0} t^j$ it follows that each unit u determines by $t \to t^* = tu$ an automorphism of F. Letting $b^* = \sum \rho_j (t^*)^j = \sum D_t^j (b^*)_{t=0} t^j$ we have, by the rules on the derivatives D_t^j,

[6] For another proof see [5, pp. 379–380].

[7] See [2].

$$
\begin{aligned}
(a+b)^* &= \sum \{D_t^j(a^*+b^*)\}_{t=0} t^j \\
&= \sum \{D_t^j a^*\}_{t=0} t^j + \sum \{D_t^j b^*\}_{t=0} t^j = a^* + b^*, \\
(ab)^* &= \sum \{D_t^j(a^* b^*)\}_{t=0} t^j \\
&= \sum_j \sum \{D_t^{j_1} a^*\}_{t=0} \{D_t^{j_2} b^*\}_{t=0} t^{j_1+j_2} = a^* b^*,
\end{aligned}
$$

where $j_1 \geqq 0$, $j_2 \geqq 0$, $j_1+j_2=j$, and similarly $(a\omega)^* = a^*\omega$.

Now let S, T be two automorphisms of G. On applying ST to the prime element t we find $t^{ST} = (t^T)^S = (tu(T))^S = t^S u(T)^S = tu(S)u(T)^S = tu(ST)$ where the u's are the units corresponding to the automorphisms. Thus the mapping $S \rightarrow u(S)$ gives rise to a crossed representation[8] of the automorphism group G in the unit group U, for $u(ST) = u(S)u(T)^S$. The latter relation may be viewed as a new multiplication $\times$ between the elements of U. We define $u_1 \times u_2$ to be $u_1 u_2^W$ where $W = S(u_1)$ is the automorphism which corresponds to u_1 by virtue of Theorem 1. The associativity of the group G implies that U is a group with respect to the operation $\times$.

We now proceed to a different interpretation of this operation. Let Ω_x be the set of all formal power series $\sum_{i=0}^{\infty} \omega_i x^i$, where x is an indeterminate over Ω.

DEFINITION 2. *If* $A = \sum \alpha_i x^i$ *and* $B = \beta_i x^i$ *then define* $A \otimes B$ *as* $\sum_i \beta_i (\sum_j \alpha_j x^j)^i$ *if and only if* $\alpha_0 = 0$.

From the definition of Ω_x we observe that Ω_x may be considered as an infinite vector space over Ω with restricted multiplication.[9] The element x plays the role of a left and right unit on the subset $\boldsymbol{U}$ of all elements without constant terms. The set $\boldsymbol{U}$ is a group, as may be verified by using Taylor expansions with respect to x. Obviously the product of elements corresponds to the operation of taking the function of a function. It is now easy to set up a relation between U with $\times$ as operation and $\boldsymbol{U}$. If $u = \sum_{j=0}^{\infty} \omega_j t^j$, $\omega_0 \neq 0$, is given then we take for the corresponding element $\boldsymbol{u} = \sum_{j=0}^{\infty} \omega_j x^{j+1}$. The product $u_1 \times u_2$ of two elements u_1, u_2 in U is then determined as $\{(\boldsymbol{u}_1 \otimes \boldsymbol{u}_2) x^{-1}\}_{x=t}$ where the factor x^{-1} is to indicate a scalar division of $\boldsymbol{u}_1 \otimes \boldsymbol{u}_2$ by x. In the sequel it will be useful to use the representation of the elements S in $\boldsymbol{G}$ by means of the elements in $\boldsymbol{U}$; $S \rightleftarrows u(S) \rightleftarrows \boldsymbol{u}(S)$. We shall use the symbol S ambiguously for the automorphism S and the representation $\boldsymbol{u}(S)$.

Suppose now that μ_1, μ_2 are two nonzero elements of Ω. These elements determine, by $t \rightarrow t\mu_i$, automorphisms $S(\mu_i)$ in $\boldsymbol{G}$, $i = 1, 2$. The

[8] See [7, p. 313].

[9] See [8, 9].

multiplication rule of U shows $S(\mu_1)S(\mu_2)=(x\mu_1\otimes x\mu_2)=S(\mu_1\mu_2)=S(\mu_2\mu_1)=(x\mu_2\otimes x\mu_1)=S(\mu_2)S(\mu_1)$. Hence the multiplicative group Ω^* of Ω has an isomorphic image in G.

The group G contains an infinite series of subsets G_i defined by the elements $S_i=x+\sum_{\nu=i+1}^{\infty}\omega_\nu x^\nu$.

THEOREM 2. *The sets G_i are invariant subgroups of G, their intersection $\bigcap_i G_i$ is the identity and each factor group G_i/G_{i+1} is isomorphic to the additive group Ω. The factor group G/G_1 is isomorphic to the multiplicative group Ω^*.*

PROOF. Let $S=x+\omega_1x^i+\cdots$, $T=x+\omega_2x^i+\cdots$ be two elements of G_i. Then, by definition of the product in U,

$$(*)\qquad \begin{aligned} S\otimes T &= x+(\omega_1+\omega_2)x^i+\cdots, \\ T\otimes S &= x+(\omega_2+\omega_1)x^i+\cdots, \\ S^{-1} &= x+(-\omega_1)x^i+\cdots. \end{aligned}$$

Thus G_i is a group. To show that G_i is an invariant subgroup of G let $W=\alpha_0x+\alpha_1x^2+\cdots$ be an arbitrary element of G. The inversion formula for power series implies $W^{-1}=\alpha_0^{-1}x+(-\alpha_1/\alpha_0^3)x^2+\cdots$. Consequently, by direct computation, $W\otimes S\otimes W^{-1}=x+\alpha_0^{i-1}\omega_1x^i+\cdots$ lies also in G_i. Definition of the G_i implies $G\supset G_1\supset\cdots$ and $\bigcap_i G_i=x$, the identity of G. The relations (*) imply that G_i/G_{i+1} is isomorphic to the additive group Ω. Now let $S(\alpha_0)$ be determined by the element $W=\alpha_0x+\alpha_1x^2+\cdots$. Then $S(\alpha_0)^{-1}\otimes W=\alpha_0(\alpha_0^{-1})x+\alpha_1(\alpha_0^{-1})x^2+\cdots=x+(\alpha_1/\alpha_0)x^2+\cdots$ lies in G_1. Hence $G=\sum_{\alpha\in\Omega^*}S(\alpha)G_1$ and therefore $G/G_1\cong\Omega^*$.

COROLLARY 1. *The group G is a group extension of G_1 by Ω^* with factor set unity.*

PROOF. As seen before the elements $S(\mu)$, μ in Ω^*, form a subgroup of G. The elements $S(\mu)=\mu x$ induce automorphisms on the group $G_1=\{x+\omega_1x^2+\cdots+\omega_nx^{n+1}+\cdots\}$. The associated combinations are determined by

$$x\mu\otimes(x+\omega_1x^2+\cdots+\omega_nx^{n+1}+\cdots)=x\mu+x^2\omega_1\mu^2+\cdots+x^{n+1}\omega_n\mu^{n+1}+\cdots,$$

$$(x+\omega_1x^2+\cdots+\omega_nx^{n+1}+\cdots)\otimes x\mu=x\mu+x^2\omega_1\mu+\cdots+x^{n+1}\omega_n\mu+\cdots,$$

whence

$$x\mu\otimes(x+\omega_1x^2+\cdots+\omega_nx^{n+1}+\cdots)\otimes(x\mu)^{-1}=x+x^2\omega_1\mu+\cdots+x^{n+1}\omega_n\mu^n+\cdots.$$

Since $\{S(\mu), \mu \in \Omega^*\}$ is a subgroup the factor set of Ω^* with respect to G is the unit factor set.

COROLLARY 2. *The group G_1 is a complete metric group; if the field Ω is finite then G_1 is compact.*

PROOF. The system of normal subgroups G_i defines a topology in G_1. The group G_1 is complete with respect to this topology for G_1 contains all elements $x+\omega_1 x^2+\cdots+\omega_n x^{n+1}+\cdots$ with arbitrary coefficients ω_n. We next define a metric in G_1 which is consistent with the system of defining neighborhoods G_i, $i=1, 2, \cdots$. Suppose $S = x + \alpha_1 x^2 + \cdots + \alpha_{n-1}x^n + \alpha_n x^{n+1} + \cdots$, $T = x + \beta_1 x^2 + \cdots + \beta_{n-1}x^n + \beta_n x^{n+1} + \cdots$ are two arbitrary elements of G_1. We define $\delta(S, T)$ to be e^{-n} if and only if $\alpha_i = \beta_i$ for all $i<n$ and $\alpha_n \neq \beta_n$. Then $\delta(S, T) = \delta(T, S)$, $\delta(S, T) > 0$ for $S \neq T$, $\delta(S, T) \leqq \max[\delta(S, R), \delta(T, R)]$. Moreover G_n consists of all elements S in G_1 which have distance $\delta(1, S) \leqq e^{-n}$. Thus G_1 is a 0-dimensional metric group. The factor groups G_1/G_n contain nontrivial elements of finite order if and only if $\chi < \infty$ holds for the characteristic of Ω. If $\chi < \infty$ the elements of G_1/G_n have at most order χ^{n-1}. Hence $\lim_{n\to\infty} g^{\chi^n} = 1$ for each element g of G_1, thus G_1 is a generalized χ-adic group.[10] In particular, G_1 is compact if Ω is finite. In this case $G_n/G_{n+1} \cong \Omega$, and thus G_1 is isomorphic to the inverse limit $\lim_{n\to\infty} G_1/G_n$ where the factor groups G_1/G_n are all finite.

DEFINITION 3. *The totality of elements S in G_1 with $a^S \equiv a \pmod{\mathfrak{P}^n}$ for each a in $\mathfrak{O}$ is called the nth pseudo-ramification group R_n of F.*[11]

THEOREM 3. *The groups G_n and R_n coincide.*

PROOF. We first remark that the first pseudo-ramification group R_1 coincides with G_1, for the general element $x+\omega_1 x^2+\cdots$ of G_1 induces the identity mapping on the residue class field Ω of F. Suppose now that $g = x + \sum_{\nu=n+1}^{\infty} \omega_\nu x^\nu$ is an arbitrary element of G_n. Let $a = \alpha_0 + \alpha_1 t + \cdots + \alpha_{n-1}t^{n-1} + \alpha_n t^n + \cdots$. Then

$$a^g = \alpha_0 + \alpha_1\left(t + \sum_{\nu=n}^{\infty} \omega_\nu t^\nu\right) + \cdots + \alpha_{n-1}\left(t + \sum_{\nu=n}^{\infty} \omega_\nu t^\nu\right)^{n-1} + \alpha_n\left(t + \sum_{\nu=n}^{\infty} \omega_\nu t^\nu\right)^n + \cdots$$

[10] A topological group G is called χ-adic, if $\lim_{n\to\infty} g^{\chi^n} = 1$ for each g in G, $\chi < \infty$, in a sense of the topology of G. Examples of χ-adic groups are the additive subgroup of the χ-adic number field and multiplicative groups in χ-adic algebras.

[11] See [6, p. 438].

$$= \alpha_0 + \alpha_1 t + \alpha_1 \sum_{\nu=n}^{\infty} \omega_\nu t^\nu + \cdots + \alpha_{n-1}t^{n-1} + \alpha_{n-1}(n-1)t^n\omega_n$$
$$+ \cdots + \alpha_n t^n + \cdots + \alpha_n n t^{n+\nu}\omega_n + \cdots$$
$$\equiv \alpha_0 + \alpha_1 t + \cdots + \alpha_{n-1}t^{n-1} \pmod{\mathfrak{P}^n},$$

whence $G_n \subseteq R_n$. Conversely, let $g \in R_n$ and $t^g = t + \beta_1 t^2 + \cdots + \beta_{n-1}t^n + \beta_n t^{n+1} + \cdots$. We compute

$$a^g = \alpha_0 + \alpha_1(t + \beta_1 t^2 + \cdots + \beta_{n-1}t^n + \beta_n t^{n+1} + \cdots)$$
$$+ \alpha_2(t + \beta_1 t^2 + \cdots)^2 + \cdots + \alpha_{n-1}(t + \beta_1 t^2 \cdots)^{n-1} + \cdots$$
$$\equiv \alpha_0 + \alpha_1 t + \alpha_1\beta_1 t^2 + \cdots + \alpha_2 t^2 + 2\alpha_2\beta_1 t^3 + \cdots + \cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+ \alpha_{n-1}t^{n-1} + \cdots$$
$$\equiv \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_{n-1}t^{n-1} \pmod{\mathfrak{P}^n}.$$

Reducing this congruence successively mod $\mathfrak{P}^2, \mathfrak{P}^3, \cdots, \mathfrak{P}^{n-1}$ we obtain $\beta_1 = \cdots = \beta_{n-2} = 0$, whence $G_n \supseteq R_n$.

REMARK. As in the ordinary Hilbert theory it follows that R_n is an invariant subgroup of G_1. This furnishes a new proof for the invariance of the groups G_n.

THEOREM 4. *Suppose that H is a finite subgroup of G_1 and let K be the associated subfield of F. The ramification groups H_i of F/K are equal to $G_i \cap H$.*

PROOF. Let V_K be the valuation which is induced by V on the subfield K. Suppose that $\Gamma(F)$ and $\Gamma(K)$ are the value groups associated to V and V_K, respectively. We first observe that Ω is contained in K for $H \subset G_1$. Let $\overline{K}$ be the completion of K with respect to V_K, then $K \subseteq \overline{K} \subseteq F$ for the fundamental sequences in K are special fundamental sequences in F by the definition of V_K. Next observe that V_K has only V for its prolongation to F. Remark that all prolongations of V_K are given as V^h, h in H; thus, by Lemma 1, $V^h = V$. Combining these facts, we find $[F:K] = [\Gamma(F):\Gamma(K)] = [\Gamma(F):\Gamma(\overline{K})] = [F:\overline{K}]$. Consequently $K = \overline{K}$, that is, K is a complete field and H is the decomposition and inertial group for F/K. Now let H_i be the ith ramification group of F/K. Then $H_i \subset R_i = G_i$ by the proof of Theorem 3. Thus $H_i \subseteq R_i \cap H$. Conversely, let S be an element of $R_i \cap H$. Then $a^S \equiv a \pmod{\mathfrak{P}^i}$ by definition of R_i. Hence S lies in H_i. Therefore $H_i = R_i \cap H$ as asserted.

Let K be an arbitrary subfield of F such that $\Omega \subset K$ and $[F:K] < \infty$. The field K is then complete with respect to the valuation V_K

induced by V. For the proof we observe that the valuation V has a unique prolongation to the normal closure of F over K, for the latter is complete since it is a finite extension of F. The ramification theory implies, as at the beginning of the proof for Theorem 4, that K is complete. Now let $\pi(F)$ be any prime element of F, that is $V\pi(F)=1$, $F=\Omega\{\pi(F)\}$. Then $N_{F/K}\pi(F)$ is a prime element $\pi(K)$ for K because $V\pi(K)=[F:K]$ and $\Omega\{\pi(K)\}=K$ by Lemma 2 applied to the chain $\Omega\{\pi(K)\}\subseteq K\subset F$. We have $\pi(K)=\sum_{\nu=0}^{\infty}\omega_\nu\pi(F)^{n+\nu}$, where $[F:K]=n$ and $\omega_0\neq 0$. Applying an automorphism S of $\mathbf{G}$ to $\pi(F)$ we obtain a field $K^S=\Omega\{\pi(K)^S\}$ which is isomorphic to K and $K=K^S$ if and only if S is an automorphism of F/K. The group SHS^{-1} is the full automorphism group of F/K^S if $\mathbf{H}$ is the automorphism group of F/K. We apply these results to prove the following theorem.

THEOREM 5. *The automorphism group* $\mathbf{G}$ *of* F *contains elements of finite order.*

PROOF. We first observe that F always has normal completely ramified extensions M. The existence of such extensions depends on the nature of the residue class field Ω.[12] In case that n is an integer prime to the characteristic χ and Ω contains a primitive nth root of unity the field F surely has at least one cyclic completely ramified extension of degree n. For example, there is surely such a quadratic extension if $(\chi, 2)=1$ or $\chi=\infty$. If χ is finite then Ω contains some field of (χ^m-1)th roots of unity, $m\geqq 1$, and any $n|(\chi^m-1)$ may be used. For finite χ there are infinitely many cyclic completely ramified extensions of degree χ over F.[13] Hence we know that completely ramified normal extensions M/F exist. Since M is completely ramified, say of degree n, we have $\pi(F)=\sum_{\nu=0}^{\infty}\omega_\nu\pi(M)^{\nu+n}$, $\omega_0\neq 0$, for a prime element $\pi(M)$ of M. By the structure theory of fields of power series the fields M and F are isomorphic.[14] Suppose that ϕ is a realization of the isomorphism $M\cong F$, that is, $M^\phi=F$. We apply ϕ to the equation for $\pi(F)$ and obtain $\pi(F)^\phi=\sum_{\nu=0}^{\infty}\omega_\nu[\pi(M)^\phi]^{n+\nu}$. Then $\pi(F)^\phi$ determines by $\Omega\{\pi(F)^\phi\}$ a subfield K of F so that the Galois groups of M/F and $F/\Omega\{\pi(F)^\phi\}$ are isomorphic. Observing that the Galois group of M/F is isomorphic to a finite subgroup of $\mathbf{G}$, the assertion of the theorem follows.[15]

As a special case we consider cyclic subgroups $\mathbf{H}$ of order n in $\mathbf{G}$ for which $(n, \chi)=1$. Let K be the field of invariants for $\mathbf{H}$. We have

[12] See [11, 14].
[13] See [14].
[14] See [1, 6, 13, 14].
[15] See [3, p. 890].

$K=\Omega\{ut^n\}$ where u is a unit of F. Then $u=\omega v$ where $\omega\neq 0$ lies in Ω and $v\equiv 1 \pmod{P}$. We have $K=\Omega\{vt^n\}$, for K is an immediate extension of $\Omega\{vt^n\}$. Since $(n, \chi)=1$ there exists[16] a unit y of F with $y\equiv 1 \pmod{P}$, $y^n=v$. Therefore $K=\Omega\{(yt)^n\}$. Since $\Omega\{yt\}=\Omega\{t\}=F$ we obtain $\boldsymbol{H}=S\boldsymbol{H}_0S^{-1}$ where S is the automorphism of $\boldsymbol{G}_1$ with $t^S=yt$ and $\boldsymbol{H}_0$ is the finite subgroup of $\boldsymbol{G}$ generated by the element ζx, ζ a primitive nth root of unity. Finally $K=\Omega\{t^n\}$ if and only if $S\boldsymbol{H}_0S^{-1}=\boldsymbol{H}_0$, that is, S induces an automorphism on $\boldsymbol{H}_0$. Expressing $K=\Omega\{t^n\}$ in terms of the prime elements we have $(yt)^n/t^n=y^n$ lies in $\Omega\{t^n\}$. Consequently y^n must be a power series $1+\sum_{j=1}^{\infty}\alpha_j(t^n)^j$. Conversely each unit y with y^n in $\Omega\{t^n\}$ gives rise to $K=\Omega\{t^n\}$. Since there are always elements y for which y^n does not lie in $\Omega\{t^n\}$ and $y^n=1+\omega t+\cdots$, $\omega\neq 0$, we have the existence of automorphisms S with $S\boldsymbol{H}_0S^{-1}\neq\boldsymbol{H}_0$ and $\Omega\{t^n\}\cap\Omega\{(yt)^n\}=\Omega$, as can be shown by comparing the series of the fields. This shows that the set of normal subfields K with $[F:K]<\infty$, $([F:K], \chi)=1$ is not a lattice. We remark that conditions on y may be derived to describe $\Omega\{(yt)^s\}=\Omega\{t^s\}$ for $s\mid n$.

The group $\boldsymbol{G}$ can be interpreted as a group of automorphisms of an infinite Lie ring $\mathfrak{O}_L$ over Ω. We define $\mathfrak{O}_L$ as the ring $\mathfrak{O}$, considered as an infinite vector space over Ω, in which a product $[f, g]$ is defined as follows. We set $[f, g]=(dg/dt)f-(df/dt)g$. The rules of differentiation imply that $[f, g]$ obeys the rules of the Jacobi bracket. For a basis of $\mathfrak{O}_L$ we may take the elements $e_i=t^{i+1}$, $i=-1, 0, 1, \cdots$ and $[e_i, e_j]=(j-i)e_{i+j}$. By actual computation it can be shown that the mapping

$$(**)\qquad f(t)\to f(t^S)/(dt^S/dt)$$

is an automorphism of $\mathfrak{O}_L$.[17] Moreover, distinct elements of $\boldsymbol{G}$ give rise to distinct automorphisms in the automorphism group $\boldsymbol{A}(\mathfrak{O}_L)$ of $\mathfrak{O}_L$.

THEOREM 6. *The groups $\boldsymbol{G}$ and $\boldsymbol{A}(\mathfrak{O}_L)$ coincide if $\chi=\infty$.*

PROOF. Suppose that Σ is an automorphism of $\mathfrak{O}_L$. Let $t^\Sigma=\Phi(t)$. We shall show that there exists an automorphism S in $\boldsymbol{G}$ so that Σ is determined by the formula (**) applied to t, that is, $t^\Sigma=t^S/(dt^S/dt)$. Such an element S determines then the automorphism Σ on all of $\mathfrak{O}_L$ because the elements t^{i+1} form a basis of $\mathfrak{O}_L$. To determine S it suffices to find a unique element $\phi(t)=\alpha_1t+\alpha_2t^2+\cdots$, $\alpha_1\neq 0$, for which $t^\Sigma=\Phi(t)=f[\phi(t)]/(d\phi/dt)$. By formal integration we find

[16] See [10, p. 561; 11, p. 441].

[17] Compare [15, pp. 37–47].

$\phi(t)=\exp[\int\Phi^{-1}(t)dt]$ where $\int\Phi^{-1}(t)dt$ is the formal indefinite integral without a constant of integration. We observe that all formal operations involved can be carried out because they are determined in $\mathfrak{O}$. The automorphism S is given by $\phi(x)$ using the representation of G by U. In concluding we remark that $G\subset A(\mathfrak{O}_L)$ for $\chi<\infty$. The inequality may be explained by the fact that t^n is never the derivative of an element in F if $n\equiv 0 \pmod{\chi}$.

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