ON UNIFORM CONVERGENCE OF TRIGONOMETRIC SERIES

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1. **Introduction.** The following theorems have been proved previously.[1]

THEOREM I. *If the function* $\phi(t)$ *is throughout continuous, periodic of period* 2π, $\phi(t)=\phi(-t)=\phi(2\pi+t)$,

$$\phi(t) \sim \frac{a_0}{2} + \sum_1^\infty a_n \cos nt, \tag{1.1}$$

and if

$$na_n > -K, \tag{1.2}$$

for some constant K, *and all* n, *then the series* (1.1) *is uniformly convergent* (*on the real axis*).

THEOREM II. *If* $f(t)$ *is everywhere continuous, periodic of period* 2π' $f(t)=-f(-t)$,

$$f(t) \sim \sum_1^\infty b_n \sin nt, \tag{1.3}$$

and if

$$nb_n > -K, \qquad n = 1, 2, 3, \cdots, \tag{1.4}$$

then the series (1.3) *is uniformly convergent.*

THEOREM III (CHAUNDY AND JOLLIFFE). *The Fourier series* (1.3) *is uniformly convergent, if*

$$b_n \geqq b_{n+1} > 0, \quad \text{and if} \quad nb_n \to 0. \tag{1.5}$$

Note that here no explicit assumption is made on $f(t)$.

THEOREM IV. *If* $\phi(t)$ *is continuous at* $t=0$, *and if*

$$\lim_{\lambda \downarrow 1} \limsup_{n\to\infty} \sum_n^{\lambda n} (|a_\nu| - a_\nu) = 0, \tag{1.6}$$

then the series (1.1) *is uniformly convergent at* $t=0$. (*That is,* $s_n(t_n)\to s$ *whenever* $t_n\to 0$.)

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[1] Cf. [2] and the references given there; numbers in brackets refer to the literature cited at the end of this paper.

THEOREM V. *If $f(t)$ is continuous at $t=0$, and if*

$$\lim_{\lambda\downarrow 1} \limsup_{n\to\infty} \sum_{n}^{\lambda n} (|b_\nu| - b_\nu) = 0, \tag{1.7}$$

then $\sum_1^n \nu b_\nu = o(n)$, and the series (1.3) *is uniformly convergent at $t=0$.*

Some more general results are given in the present paper. In particular:

THEOREM 1. *Under the assumptions of Theorem* IV *the series* (1.1) *converges uniformly at each point of continuity of $\phi(t)$.*

THEOREM 2. *Under the assumptions of Theorem* V *the series* (1.3) *converges uniformly at each point of continuity of $f(t)$.*

Clearly Theorems 1 and 2 include Theorems I and II respectively. Either of the following two theorems includes Theorem III.

THEOREM 3. *Suppose that*

$$\sum_{n}^{2n} |b_\nu - b_{\nu+1}| = O(n^{-1}) \quad \text{as} \quad n\to\infty, \tag{1.8}$$

and that

$$(1-r)\sum_{1}^{\infty} nb_n r^n \to 0 \quad \text{as} \quad r\uparrow 1; \tag{1.9}$$

then the trigonometric series $\sum b_n$ sin *nt is uniformly convergent.*

Note that the assumptions refer solely to the coefficients b_n.

THEOREM 4. *Suppose, for some constants $p \geqq 0$, $q \geqq 0$,*

$$nb_n + p = B_n \geqq 0, \tag{1.10}$$

that

$$B_{n+1} \leqq (1 + n^{-1}q)B_n, \quad \textit{for all large } n, \tag{1.11}$$

and that (1.9) *holds. Then $nb_n \to 0$ and the trigonometric series $\sum b_n$* sin *nt is uniformly convergent.*

We also give (in §§5 and 6) analogous theorems for cosine series; here the partial sums $\sum_1^n a_\nu = s_n$ play a similar role as the sequence $\{nb_n\}$ for the sine series. However convergence of the series $\sum a_\nu$ does not carry as far as existence of the limit lim nb_n. It is for this reason that no such theorems have been established hitherto for cosine series. For details see §§5, 6 and 7.

2. **Proof of Theorems 1 and 2.** We have proved [2] that under the assumptions of Theorem IV

$$\lim_{\lambda\downarrow 1}\ \limsup_{n\to\infty}\ \sum_{n}^{\lambda n}|a_\nu| = 0. \tag{2.1}$$

If $\phi(t)$ is continuous at t_0, then the Fourier series

$$\frac{\phi(t_0+\theta)+\phi(t_0-\theta)}{2} \sim \sum a_n \cos nt_0 \cos n\theta,$$

$$\frac{\phi(t_0+\theta)-\phi(t_0-\theta)}{2} \sim \sum a_n \sin nt_0 \sin n\theta$$

satisfy the assumptions of Theorems IV and V respectively, hence are uniformly convergent at $\theta=0$. This proves Theorem 1. The proof of Theorem 2 follows on quite similar lines, since it has been proved [2] that

$$\lim_{\lambda\downarrow 1}\ \limsup_{n\to\infty}\ \sum_{n}^{\lambda n}|b_\nu| = 0. \tag{2.2}$$

It is clear from our proof that the assumptions of our theorems can be replaced by the sole assumptions (2.1) and (2.2) respectively.

We remark that in Theorems IV and V the assumptions (1.6) and (1.7) cannot be replaced by

$$\sum_{n}^{2n}|a_\nu| = O(1) \quad \text{and} \quad \sum a_n \text{ converges},$$

$$\sum_{n}^{2n}|b_\nu| = O(1) \quad \text{and} \quad \sum_{1}^{n} \nu b_\nu = o(n),$$

respectively. We give an example, suggested by a construction due to Fejér [1].

Let $P_n(z)=\sum_{\nu=0}^{n-1}z^\nu/(n-\nu)-\sum_{\nu=0}^{n-1}z^{n+\nu}/(\nu+1)$, then $|P_n(z)|<6$ for $|z|\leqq 1$. Let $\mu_n=2^{n^2}$, $\kappa_n=2^{n(n+1)}$, $n=1, 2, 3, \cdots$, and consider the polynomial series $\sum_1^\infty n^{-2}z^{\mu_n}P_{\kappa_n}(ze^{i/n})$. This series is clearly uniformly convergent for $|z|\leqq 1$, the degree of the nth term is $2\kappa_n+\mu_n-1<\mu_{n+1}$, hence writing out the polynomials successively we get a power series, convergent for $|z|<1: \sum_1^\infty c_nz^n=F(z)$, and $F(e^{it})$ is the Fourier power series of a continuous function. The structure of P_n easily yields $\sum_n^{2n}|c_\nu|=O(1)$. It can be proved, as in Fejér's example, that the series $\sum c_ne^{int}$ converges for each t, uniformly in $\epsilon\leqq t\leqq 2\pi-\epsilon$, $\epsilon>0$; but neither component converges uniformly at $t=0$. The same is true for the series $\sum a_n \cos nt$, $\sum a_n \sin nt$, where $a_n = R(c_n)$; $\sum a_n$ converges, so that $\sum_1^n \nu a_\nu = o(n)$. Again, using Fejér's device, and replacing $e^{i/n}$ by e^{it_n}, where the sequence $\{t_n\}$ is everywhere dense in $(0, 2\pi)$, we get a continuous function with a Fourier series and its conjugate nonuniformly convergent everywhere, while $|c_n|$ is the same as before.

3. **Proof of Theorem 3.** It follows from (1.8) that $\lim b_n$ exists, and now from (1.9) that $\lim b_n = 0$. Furthermore

$$\sum_1^{2^\kappa} |b_\nu - b_{\nu+1}| \leqq \sum_{n=0}^{\kappa-1} \sum_{2^n}^{2^{n+1}} |b_\nu - b_{\nu+1}| = \sum \frac{1}{2^n} O(1) = O(1),$$

hence

$$\sum_1^\infty |b_\nu - b_{\nu+1}| < \infty. \tag{3.1}$$

Moreover

$$\begin{aligned} \sum_n^\infty |b_\nu - b_{\nu+1}| &\leqq \sum_{\kappa=0}^\infty \sum_{n\cdot 2^\kappa}^{n\cdot 2^{\kappa+1}} |b_\nu - b_{\nu+1}| \\ &= O\left(\frac{1}{n} \sum \frac{1}{2^\kappa}\right) = O\left(\frac{1}{n}\right), \end{aligned} \tag{3.2}$$

hence

$$nb_n = n \sum_n^\infty (b_\nu - b_{\nu+1}) = O(1). \tag{3.3}$$

It was proved by Littlewood that boundedness of a sequence and Abel summability imply $(C, 1)$ summability; if we apply this to the sequence $\{nb_n\}$ it follows from (1.9) and (3.3) that

$$\sum_1^n \nu b_\nu = o(n). \tag{3.4}$$

Next, from Abel's formula

$$\sum_n^m b_\nu \sin \nu t = \sum_n^{m-1} (b_\nu - b_{\nu+1}) T_\nu(t) + b_m T_m(t) - b_n T_{n-1}(t), \tag{3.5}$$

where

$$T_n(t) = \frac{\cos t/2 - \cos (n + 1/2)t}{2 \sin t/2},$$

hence in any interval $\epsilon \leqq t \leqq 2\pi - \epsilon$

$$\left|\sum_n^m b_\nu \sin \nu t\right| < \epsilon^{-1}\pi \sum_n^\infty |b_\nu - b_{\nu+1}| + 2\epsilon^{-1}\pi(|b_n| + |b_m|).$$

Thus the series $\sum b_n \sin nt$ is uniformly convergent in $\epsilon \leqq t \leqq 2\pi - \epsilon$, $\epsilon > 0$. Let

$$\sum_1^\infty b_n \sin nt = f(t);$$

we shall prove next that $f(t) \to 0$ as $t \downarrow 0$. We write

$$f(t) = \left(\sum_1^n + \sum_{n+1}^\infty\right) b_\nu \sin \nu t = U_1(t) + U_2(t),$$

say, where $n = [\epsilon^{-1} t^{-1}]$. Now, employing (3.2), (3.3) and (3.5)

$$\begin{aligned} |U_2(t)| &< t^{-1}\pi\left(\sum_{n+1}^\infty |b_\nu - b_{\nu+1}| + |b_{n+1}|\right) \\ &= t^{-1}O(n^{-1}) = \epsilon O(1). \end{aligned} \tag{3.6}$$

As to $U_1(t)$, we have

$$U_1 = \sum_1^n \nu b_\nu \frac{\sin \nu t}{\nu} = \sum_1^{n-1} v_\nu \Delta_\nu + v_n \frac{\sin nt}{n},$$

where

$$v_n = \sum_1^n \nu b_\nu, \qquad \Delta_n = \Delta \frac{\sin nt}{n} = \frac{\sin nt}{n} - \frac{\sin (n+1)t}{n+1}.$$

We have

$$\Delta_n = \int_0^t (\Delta \cos nx) dx = R \int_0^t z^n (1 - z) dx, \qquad z = e^{ix},$$

hence

$$|\Delta_n| < \int_0^t |1 - z| dx < t^2,$$

and

$$\begin{aligned} |U_1(t)| &< t^2 \sum_1^n |v_\nu| + n^{-1}|v_n| \\ &< \epsilon^{-2} n^{-1} \sum_1^n \nu^{-1} |v_\nu| + n^{-1}|v_n| \to 0 \end{aligned} \tag{3.7}$$

as $t \downarrow 0$, by (3.4).

Now (3.6) and (3.7) yield

$$\limsup_{t\to 0} |f(t)| \leqq \epsilon;$$

ϵ being arbitrary, we get $f(t)\to 0$ as $t\to 0$. In view of (3.3) uniform convergence now follows from Theorem II.

We remark that under the assumptions of Theorem 3 the sequence $\{nb_n\}$ need not have a limit. This is seen from the example

$$nb_n = 1 \quad \text{for} \quad n = 2^\nu, \ \nu = 0, 1, 2, \cdots, \quad b_n = 0 \text{ otherwise.}$$

Moreover in this case $b_n \geqq 0$ and $\sum b_n$ is convergent.

On the other hand for the example $\sum_2^\infty (-1)^n \sin (2n-1)t/n \log n$, $nb_n\to 0$, $\sum b_n$ converges, yet the series is divergent for $t=\pi/2$. Of course (1.8) is not satisfied, but $\sum_n^{2n} |b_\nu| = O(1/\log n)$.

4. **Proof of Theorem 4.** We shall employ the following lemma.

LEMMA 1. *Suppose that $B_n \geqq 0$, that for some $q \geqq 0$*

$$(4.1) \qquad B_{n+1} \leqq (1 + q/n)B_n, \qquad n = 1, 2, 3, \cdots,$$

and that the sequence $\{B_n\}$ is Abel summable to B; then $B_n\to B$.

This is Lemma 5 of my paper [2]. Note that the inequalities $B_n \geqq 0$ and (4.1) need only be satisfied for all large n, $n \geqq n_0$, say. For the sequence $B_n' = B_{n_0}$, $n = 1, 2, \cdots, n_0$, $B_n' = B_n$, $n > n_0$, satisfies the assumptions of the lemma, hence $\lim B_n = \lim B_n'$ exists.

Now for $nb_n + p = B_n$, from (1.9)

$$(4.2) \qquad (1 - r)\sum B_n r^n \to p \quad \text{as} \quad r \uparrow 1;$$

from (1.10) and (1.11)

$$(4.3) \qquad 0 \leqq B_{n+1} \leqq (1 + q/n)B_n, \quad \text{for all large } n.$$

Lemma 1 now yields

$$(4.4) \qquad B_n \to p, \quad \text{that is} \quad nb_n \to 0.$$

From (4.3)

$$(4.5) \qquad (B_{n+1} - B_n) \leqq n^{-1}qB_n, \quad \text{for } n \geqq n_0, \text{ say.}$$

Write $\sum_n^{2n}(B_{\nu+1} - B_\nu) = \sum' + \sum''$, where $\sum'$ is the sum of the positive terms, and $\sum''$ the rest. From (4.4) and (4.5), $\sum' = O(1)$; furthermore

$$B_{2n+1} - B_n = \sum' + \sum'' = \sum' - \left|\sum''\right|,$$

hence $\left|\sum''\right| = B_n - B_{2n+1} + \sum' = O(1)$. It now follows that

$$\sum_{n}^{2n} | B_{\nu+1} - B_\nu | = \sum{}' + \left| \sum{}'' \right| = O(1);$$

this and (4.4) yield (1.8). Our theorem now follows from Theorem 3.

If we replace (1.9) by the assumption (A) $\lim nb_n = \rho$, then the trigonometric series

$$\sum (b_n - \rho n^{-1}) \sin nt = \sum \beta_n \sin nt$$

satisfies the assumptions of Theorem 4, hence it is uniformly convergent, and we get $nb_n \to \rho$, and

$$\sum b_n \sin nt \to \pi\rho/2 \quad \text{as} \quad t \downarrow 0. \tag{4.6}$$

Combined with Theorem 3 of our paper [2] we get the theorem.

THEOREM 5. *If* (4.2) *holds then a necessary and sufficient condition that* (4.6) *holds is* $nb_n \to \rho$.

For b_n positive and decreasing, $\rho = 0$, the result is due to Chaundy and Jolliffe, for $\rho \neq 0$ to Hardy. For references see [2].

5. **The cosine series.** We shall next prove the theorem:

THEOREM 6. *Suppose that*

$$\sum_{n}^{2n} | a_\nu - a_{\nu+1} | = O(n^{-1}), \tag{5.1}$$

and that $\sum a_n$ *is Abel summable, then* $\sum a_n \cos nt$ *is uniformly convergent.*

Using Abel's formula

$$\sum_{n}^{m} a_\nu \cos \nu t = \sum_{n}^{m-1} (a_\nu - a_{\nu+1}) \gamma_\nu(t) + a_m \gamma_m(t) - a_n \gamma_{n-1}(t), \tag{5.2}$$

where

$$\gamma_n(t) = \frac{\sin (n + 1/2)t}{2 \sin (t/2)}.$$

As in §3 it follows from (5.1) that $\lim a_n$ exists, and now Abel summability of $\sum a_n$ implies that $a_n \to 0$. Furthermore

$$\sum_{1}^{\infty} | a_n - a_{n+1} | < \infty, \quad \sum_{n}^{\infty} | a_\nu - a_{\nu+1} | = O(n^{-1}), \quad na_n = O(1). \tag{5.3}$$

Hence, by a theorem of Littlewood, $\sum a_n$ converges.

Now (5.2) yields uniform convergence of $\sum a_n \cos nt$ in $\epsilon \leqq t \leqq \pi$, $\epsilon > 0$. Let

$$\sum_1^\infty a_n \cos nt = \phi(t), \qquad 0 < t \leqq \pi.$$

We write

$$\sum_1^\infty a_\nu \cos \nu t = \sum_1^n + \sum_{n+1}^\infty = V_1(t) + V_2(t),$$

say, where $n = [\epsilon^{-1}t^{-1}]$. Now from (5.2)

$$V_2(t) = \sum_{n+1}^\infty (a_\nu - a_{\nu+1})\gamma_\nu(t) - a_{n+1}\gamma_n(t),$$

hence

$$\begin{aligned} |V_2(t)| &< t^{-1}\pi\left(|a_{n+1}| + \sum_{n+1}^\infty |a_\nu - a_{\nu+1}|\right) \\ &= t^{-1}O(n^{-1}) = \epsilon O(1). \end{aligned} \tag{5.4}$$

To estimate V_1 put $\sum_n^\infty a_\nu = r_n$, then $r_1 = s$, and

$$V_1 = s \cos t - r_{n+1} \cos nt + 2\sum_2^n r_\nu \sin (t/2) \sin (\nu + 1/2)t,$$

hence

$$\begin{aligned} |V_1(t) - s \cos t| &\leqq |r_{n+1}| + t\sum_2^n |r_\nu| \\ &\leqq |r_{n+1}| + \epsilon^{-1}n^{-1}\sum_2^n |r_\nu| = \epsilon^{-1}o(1) \end{aligned} \tag{5.5}$$

as $t \to 0$. From (5.4) and (5.5) $\limsup_{t\to 0} |\phi(t) - s| \leqq \epsilon$, ϵ being arbitrary, we get $\phi(t) \to s$, as $t \to 0$. Our theorem now follows from Theorem I. The example $\sum 2^{-n} \cos 2^n t$ shows that na_n need not have a limit.

Here is an alternative proof for the continuity of $\phi(t)$ at $t = 0$: From (5.2)

$$\begin{aligned} \phi(t) = &- a_1/2 + 2^{-1}\sum_1^\infty (a_n - a_{n+1}) \cos nt \\ &+ 2^{-1} \cos (t/2) \sum_1^\infty (a_n - a_{n+1}) \frac{\sin nt}{\sin (t/2)}; \end{aligned}$$

clearly $\sum (a_n - a_{n+1}) \cos nt$ is uniformly convergent. Furthermore

$$\sum_1^\infty (a_n - a_{n+1}) \sin nt = \sum_1^\infty n(a_n - a_{n+1}) \frac{\sin nt}{n} = \sum_1^\infty a_n' \frac{\sin nt}{n},$$

where

$$a_n' = n(a_n - a_{n+1}), \qquad \sum_n^{2n} | a_\nu' | = O(1), \quad \text{by (5.1).}$$

Now $\sum_1^n a_\nu' = \sum_1^{n+1} a_\nu - (n+1)a_{n+1}$; $\sum a_n$ being convergent, it follows that $n^{-1}\sum_1^n \nu a_\nu \to 0$, and $\sum a_n'$ is $(C, 1)$ summable to s, hence by Theorem 4 of our paper [3]

$$\sum_1^\infty a_n' \frac{\sin nt}{nt} \to \sum_1^\infty a_n = s, \quad \text{as} \quad t \to 0.$$

Thus $\phi(t)$ is continuous at $t=0$.

Theorems 3 and 6 combined yield the theorem:

THEOREM 7. *Suppose that*

$$\sum_n^{2n} | c_\nu - c_{\nu+1} | = O(n^{-1}) \quad \textit{as} \quad n \to \infty,$$

and that $\sum c_n$ *is Abel summable; then the power series* $\sum c_n z^n$ *is uniformly convergent in the circle* $|z| \leqq 1$.

It suffices to consider the circle $|z| = 1$; suppose first that the c_n are real. The uniform convergence of $\sum c_n \cos nt$ follows from Theorem 6; it also follows that $n^{-1}\sum_1^n \nu c_\nu \to 0$, and Theorem 3 now yields the uniform convergence of $\sum c_n \sin nt$. If the c_n are complex, $c_n = a_n + ib_n$, then apply the result just obtained to $\sum a_n z^n$, $\sum b_n z^n$. This proves Theorem 7.

6. Further theorems on cosine series.

Our next theorem is:

THEOREM 8. *Suppose that for some constants* $p \geqq 0$ *and* $q \geqq 0$

$$(6.1) \quad 0 \leqq (n+1)s_{n+1} - ns_n + p \leqq (1 + q/n)[ns_n - (n-1)s_{n-1} + p],$$

$s_n = \sum_1^n a_\nu$, *and that* $\sum a_n$ *is Abel summable; then* $na_n \to 0$, *and* $\sum a_n \cos nt$ *is uniformly convergent.*

Put $ns_n - (n-1)s_{n-1} + p = \delta_n = s_n + (n-1)a_n + p$, $s_0 = 0$, then $\sum_1^n \delta_\nu = n(s_n + p) \geqq 0$, $s_n \geqq -p$, hence by a well known theorem of Tauberian type $\sum a_n$ is $(C, 1)$ summable, thus the sequence $\{\delta_n\}$ is $(C, 2)$ summable. This and $0 \leqq \delta_{n+1} \leqq (1 + q/n)\delta_n$ imply by Lemma 1 that $\lim \delta_n$ exists, $\delta_n \to \delta$, say. It follows that $n^{-1}\sum_1^n \delta_\nu = s_n + p \to \delta$, or $s_n \to \delta - p = s$, and now

$$(6.2) \qquad na_n = \delta_n - s_n + a_n - p \to 0.$$

Next from (6.1)

$$\delta_{n+1} - \delta_n \leqq (q/n)\delta_n, \tag{6.3}$$

furthermore

$$D_n = \sum_n^{2n} (\delta_{\nu+1} - \delta_\nu) = \delta_{2n+1} - \delta_n = O(1). \tag{6.4}$$

Write $D_n = D' + D''$, where D' denotes the sum of positive terms, $D'' = D_n - D'$. From (6.3)

$$0 \leqq D' \leqq q \sum_n^{2n} \nu^{-1}\delta_\nu = O(1),$$

and now from (6.4), $|D''| = O(1)$, hence

$$\sum_n^{2n} |\delta_{\nu+1} - \delta_\nu| = O(1).$$

Also $\delta_{\nu+1} - \delta_\nu = (\nu+1)(a_{\nu+1} - a_\nu) + 2a_\nu$, thus

$$\sum_n^{2n} \nu |a_{\nu+1} - a_\nu| \leqq O(1) + 2\sum_n^{2n} |a_\nu|.$$

But from (6.2), $\sum_n^{2n} |a_\nu| = O(1)$, hence

$$\sum_n^{2n} |a_{\nu+1} - a_\nu| = O(n^{-1}).$$

Our theorem now follows from Theorem 6.

We next prove the following analogue to Lemma 1:

LEMMA 2. *Suppose that* $B_n \geqq 0$ *for* $n > n_0$, *that for some* $q > 0$

$$B_{n+1} \geqq (1 - q/n)B_n, \qquad n > n_0, \tag{6.5}$$

and that (A) $\lim B_n = B$; *then* $B_n \to B$.

We may assume that $q/n < 1$ for $n > n_0$; then from (6.5)

$$\sum_n^{n+\kappa} B_\nu \geqq B_n \sum_{\nu=0}^{\kappa} (1 - q/n)^\nu = \frac{nB_n}{q}\left\{1 - (1 - q/n)^{\kappa+1}\right\}, \qquad n > n_0,$$

hence

$$\begin{aligned} &B_n \leqq \frac{q}{n}(B'_{n+\kappa} - B'_{n-1})\left\{1 - (1 - q/n)^{\kappa+1}\right\}^{-1}, \quad \text{where} \\ &B'_n = \sum_1^n B_\nu. \end{aligned} \tag{6.6}$$

Choose $\kappa=[\delta n]$, where $\delta>0$; from Abel summability and from $B_n\geqq 0$, it follows that $n^{-1}B_n'\to B$. Now from (6.6)

$$\limsup B_n \leqq \frac{q\delta B}{1-\exp(-q\delta)};$$

letting $\delta\downarrow 0$, we get

$$\limsup B_n \leqq B.$$

Similarly for $n-\kappa>n_0$

$$\sum_{\nu=0}^{\kappa} B_{n-\nu} \leqq B_n \sum_{\nu=0}^{\kappa}\left(1-\frac{q}{n-\kappa}\right)^{-\nu}$$
$$= \frac{n-\kappa}{q} B_n\left(1-\frac{q}{n-\kappa}\right)\left\{\left(1-\frac{q}{n-\kappa}\right)^{-(\kappa+1)}-1\right\},$$

hence

$$B_n \geqq (q/(n-\kappa-q))(B_n'-B_{n-\kappa-1}')\{(1-q/(n-\kappa))^{-(\kappa+1)}-1\}^{-1}.$$

Let now $\kappa=[n\delta]$, $0<\delta<1$, then

$$\liminf B_n \geqq \frac{q\delta}{1-\delta}B\left(\exp\frac{q\delta}{1-\delta}-1\right)^{-1},$$

and $\delta\downarrow 0$ yields $\liminf B_n\geqq B$. This proves the lemma.

THEOREM 9. *Suppose that for some constants* $p\geqq 0$, $q\geqq 0$,

$$(6.7)\qquad \begin{aligned}&(n+1)s_{n+1}-ns_n+p\\ &\qquad\geqq(1-q/n)[ns_n-(n-1)s_{n-1}+p]\geqq 0,\end{aligned}$$

and that (A) $\lim s_n=s$ *exists. Then* $na_n\to 0$ *and* $\sum a_n\cos nt$ *is uniformly convergent.*

As in the proof of Theorem 8, $s_n\geqq -p$, hence $\sum a_n$ is $(C,1)$ summable; then by Lemma 2, $\delta_n\to\delta$, $na_n\to 0$. Next from (6.7)

$$\delta_{n+1}-\delta_n \geqq -qn^{-1}\delta_n, \quad \text{and}$$

$$(6.8)\qquad D_n=\sum_{n}^{2n}(\delta_{\nu+1}-\delta_\nu)=\delta_{2n+1}-\delta_n=O(1).$$

Write $D_n=D'+D''$, where D' denotes the sum of negative terms, $D''=D_n-D'$. From (6.8), $0\geqq D'\geqq -q\sum_n^{2n}\nu^{-1}\delta_\nu$, hence $D'=O(1)$, and $D''=O(1)$; hence $\sum_n^{2n}|\delta_{\nu+1}-\delta_\nu|=O(1)$. The remaining part is the same as in the proof of Theorem 8.

7. Closing remarks. The assumption of Lemma 1 can be written as $0 \leqq B_{n+1} \leqq (n+q)/nB_n$, or $0 \leqq (\Gamma(n+q)/\Gamma(n))B_{n+1} \leqq (\Gamma(n+q+1)/\Gamma(n+1))B_n$, that is, $\Gamma(n)B_n/\Gamma(n+q)$ is decreasing. A similar lemma was proved by Hardy; for reference see [2]. Again in Lemma 2 the assumption is $B_{n+1} \geqq (n-q)/nB_n \geqq 0$, or

$$(\Gamma(n-q)/\Gamma(n))B_{n+1} \geqq (\Gamma(n-q+1)/\Gamma(n+1))B_n \geqq 0,$$

that is, $\Gamma(n)B_n/\Gamma(n-q)$ is increasing. The larger q the more general is the condition.

The differences $(n+1)s_{n+1} - ns_n = \tau_{n+1}$ are the $(C, -1)$ means of the series $\sum a_n$, that is, $s_n = n^{-1}\sum_1^n \tau_\nu$ $(\tau_1 = s_1)$. The condition (6.1) may be written as

$$-(\tau_n + p) \leqq \tau_{n+1} - \tau_n \leqq (q/n)(\tau_n + p).$$

If it holds for some p, then it clearly holds for any $p' > p$. Similarly (6.7) becomes

$$\tau_{n+1} - \tau_n \geqq -(q/n)(\tau_n + p) \geqq -(\tau_n + p),$$

and here too p may be replaced by any $p' > p$. Clearly summability $(C, -1)$ of the series $\sum a_n$ is equivalent to convergence together with $na_n \to 0$.

We have seen that the first inequality of (6.1) and Abel summability of $\sum a_n$ imply $(C, 2)$ summability of the sequence $\{\tau_n\}$; it follows from a theorem of Tauberian type that $\sum a_n$ converges. It is an open question whether this and $\tau_n \geqq -p$, $n = 1, 2, 3, \cdots$, imply uniform convergence of $\sum a_n \cos nt$ at $t = 0$. Theorem IV asserts that this is the case if $\sum a_n \cos nt$ is the Fourier series of a function continuous at $t = 0$. However it is doubtful whether even $(C, -1)$ summability of $\sum a_n$ itself implies uniform convergence of $\sum a_n \cos nt$, or continuity of the corresponding function at $t = 0$.

References

1. L. Fejér, *Über Potenzreihen, deren Summe im abgeschlossenen Konvergenzkreise überall stetig ist*, Sitzungsberichte der Kgl. Bayerischen Akademie der Wissenschaften, 1917, pp. 33–50.

2. O. Szász, *On uniform convergence of Fourier series*, Bull. Amer. Math. Soc. vol. 50 (1944) pp. 587–595.

3. ———, *On Abel and Lebesgue summability*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 885–893.

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