# TWO NOTES ON MEASURE THEORY

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I. In a recent paper [1],[1] Saks has indicated a construction whereby a Carathéodory outer measure can be produced on any compact metric space $M$, provided that a certain linear functional $\Phi$ is defined on the set $\mathfrak{C}$ of all continuous real-valued functions whose domain is $M$. The functional $\Phi$ is required to be non-negative for non-negative functions, and to have the property that if the sequence $\{f_n\}$ has the uniform limit 0, then the sequence $\Phi(f_n)$ is a null-sequence. (The measure itself can be defined without this last property.) The purpose of this note is to show that such a linear functional always exists, in a non-trivial form, specifically, so that $\Phi(1)=1$.

We consider the set $\mathfrak{C}$ as a linear space, and together with $\mathfrak{C}$ the linear space $\mathfrak{K}\subset\mathfrak{C}$, where $\mathfrak{K}$ consists of all constant functions. On the entire space $\mathfrak{C}$, we define a functional $p(f)=\sup_{x\in M} f(x)$. This least upper bound always exists, since $M$, being a compact metric space, is a bicompact space, on which every continuous real-valued function is bounded. It is easy to verify that $p(f+g)\leqq p(f)+p(g)$, for all $f$, $g\in\mathfrak{C}$, and that $p(tf)=tp(f)$ whenever $t$ is a non-negative real number. We define a linear functional $\Phi$ on the subspace $\mathfrak{K}$ as follows: $\Phi(f)=f(x)$ for an arbitrary $x\in M$. It is clear that $\Phi(f)=p(f)$ for $f\in\mathfrak{K}$ and that $\Phi$ is linear on $\mathfrak{K}$. By virtue of the celebrated theorem of Hahn-Banach, it appears that $\Phi$ can be extended linearly to all of $\mathfrak{C}$ in such a fashion that $\Phi(f)\leqq p(f)$ for all $f\in\mathfrak{C}$. We further observe that $\Phi$ may be taken non-negative for non-negative functions. For, if $\Phi$ has been defined by the Hahn-Banach construction for all $f\in\mathfrak{B}$, where $\mathfrak{K}\subset\mathfrak{B}\subset\mathfrak{C}$, $\mathfrak{B}\neq\mathfrak{C}$, and if $g\in\mathfrak{C}-\mathfrak{B}$ and $g\geqq 0$, then the number $a=\inf_{f\in\mathfrak{B}}(p(f+g)-\Phi(f))$ is an upper bound to possible values for $\Phi(g)$. $a$, however, is plainly non-negative, so that $\Phi(g)$ may always be taken non-negative. Suppose now that the sequence of functions $\{f_n\}$ has the uniform limit 0. The function $\epsilon-f_n$ is non-negative for all $n>N(\epsilon)$, $N(\epsilon)$ being some natural number dependent upon the arbitrary positive number $\epsilon$. Accordingly, $\Phi(\epsilon-f_n)=\Phi(\epsilon)-\Phi(f_n)=\epsilon\Phi(1)-\Phi(f_n)=\epsilon-\Phi(f_n)\geqq 0$. Likewise, it is easy to show that $\epsilon+\Phi(f_n)\geqq 0$ for all sufficiently large $n$. It follows at once that $\lim_{n\to\infty}\Phi(f_n)=0$. It is proved in Saks [1] that the functional $\Phi$ can

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[1] Numbers in brackets refer to correspondingly numbered articles in the bibliography at the end of the paper.

be used to define a Carathéodory outer measure under which every Borel set is measurable.

II. The present note has as its object the proof of the following result.

THEOREM. *If $E$ is any infinite set, there exists a non-negative real-valued function $\Gamma$ defined on all subsets of $E$ such that:*

(1) $\Gamma(A\cup B)\leqq\Gamma(A)+\Gamma(B)$;
(2) $\Gamma(A)\leqq\Gamma(B)$ *if* $A\subset B$;
(3) $\Gamma(\sum_{n=1}^{\infty}A_n)=\sum_{n=1}^{\infty}\Gamma(A_n)$ *if* $A_n\cap A_m=0$ *for* $m\neq n$;
(4) $\Gamma(E)=1$;
(5) $\Gamma(0)=0$;
(6) *the function* $\Gamma$ *assumes an infinite number of different values*;
(7) $\Gamma(\{p\})=0$ *for all points* $p\in E$ *except for those in a countable subset of* $E$.

Since Ulam has proved that a function enjoying properties (4) and (3) cannot vanish for all subsets containing exactly one point (where $E$ has any of a wide class of cardinal numbers), it appears that the present theorem is the strongest result possible.

The proof of this theorem depends upon a consideration of the family $\mathfrak{B}$ of all bounded real-valued functions defined on the set $E$. As in the preceding note, it is easy to prove the existence of a linear functional $\Phi$ defined on the family $\mathfrak{B}$ considered as a linear space. The construction, for our present purposes, will be considered in more detail. Let $E$ be partitioned into $\aleph_0$ disjoint sets $E_1, E_2, E_3, \cdots, E_n, \cdots$, each having cardinal number equal to the cardinal number of $E$. Let $\omega_n$ be the characteristic function of the set $E_n$. It is obvious that $\omega_n\in\mathfrak{B}$ for every $n$. We shall first define the linear functional $\Phi$ on the linear spaces $\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3, \cdots, \mathfrak{P}_n, \cdots$ obtained from $\mathfrak{K}$, the space of constant functions, by adjoining $\omega_1, \omega_2, \omega_3, \cdots, \omega_n, \cdots$ in succession and forming all possible linear combinations. As in the preceding note, we define $p(f)$ as $\sup_{x\in E} f(x)$, and $\Phi(f)$ as $f(x)$ for $f\in\mathfrak{K}$, $x$ being any point of $E$. By the Hahn-Banach construction, if $\Phi(f)$ is to be bounded by $p(f)$, we must have, when we calculate $\Phi(\omega_1)$, $a_1\leqq\Phi(\omega_1)\leqq b_1$, where $a_1=\sup_{f\in\mathfrak{K}}\ (-p(-f-\omega_1)-\Phi(f))$ and $b_1=\inf_{f\in\mathfrak{K}}\ (p(f+\omega_1)-\Phi(f))$. It is easy to show that $b_1=1$ and that $a_1=0$. We may, then, in accordance with the Hahn-Banach construction, take $\Phi(\omega_1)$ as $1/2$.

The numbers $a_2=\sup_{f\in\mathfrak{P}_1}\ (-p(-f-\omega_2)-\Phi(f))$ and $b_2=\inf_{f\in\mathfrak{P}_1}\ (p(f+\omega_2)-\Phi(f))$ are lower and upper bounds, respectively, for $\Phi(\omega_2)$. $a_2$ may be computed as 0, and $b_2$, as it is easy to see, is equal to 1/2. We may thus put $\Phi(\omega_2)=1/4$. This process may be continued by finite induction; it is found that the function $\omega_n$ may be assigned the value $1/2^n$ under the functional $\Phi$. $\Phi$ having been defined for the linear space generated by $\mathfrak{K}$ and the sequence $\{\omega_n\}$, the Hahn-Banach construction is carried out for the rest of $\mathfrak{B}$ in any fashion consonant with the restrictions of that theorem, provided that $\Phi(f)\geqq 0$ for non-negative functions $f$. We thus have a linear non-negative functional defined on all of the space $\mathfrak{B}$.

The measure $\Gamma(A)$ for every subset $A$ of $E$ can now be defined: $\Gamma(A)=\Phi(\omega_A)$, $\omega_A$ being the characteristic function of the set $A$. Properties (1)–(7) can now be established. It is plain that $\omega_{A\cup B}\leqq\omega_A+\omega_B$, whence $\Phi(\omega_A+\omega_B-\omega_{A\cup B})\geqq 0$, and consequently $\Phi(\omega_{A\cup B})\leqq\Phi(\omega_A)+\Phi(\omega_B)$, which inequality establishes (1). It is also obvious that $A\subset B$ implies that $\omega_A\leqq\omega_B$. From this, we infer property (2).

We examine (3) in some detail. If $A$ and $B$ are disjoint sets, it follows that $\omega_{A\cup B}=\omega_A+\omega_B$, and consequently $\Phi(\omega_{A\cup B})=\Phi(\omega_A)+\Phi(\omega_B)$, that is, the measure is additive for all subsets of $E$. We may thus state that all subsets of $E$ are measurable in the sense of Carathéodory. It is easy to prove from this fact that if $\{A_n\}$ is any sequence of pairwise disjoint sets, then $\Gamma(\sum_{n=1}^{\infty}A_n)=\sum_{n=1}^{\infty}\Gamma(A_n)$. The proof may be carried over word for word from a similar proof in Saks [2, chap. 2, §4, p. 44, Theorem 4.1].

Statements (4), (5), and (6) are immediate consequences of the definitions of $\Phi$ and $\Gamma$.

To prove that $\Gamma$ vanishes for all points except those in a countable subset, we assume the contrary. If an uncountable set $T$ of points exist such that $\Gamma(\{p\})>0$ for every $p\in T$, then there is some $\epsilon>0$ with $\Gamma(\{p_n\})>\epsilon$, where $p_n\in T$, and $n=1, 2, 3, \cdots$. On account of property (3), we have $\Gamma(\sum_{n=1}^{\infty}\{p_n\})=\sum_{n=1}^{\infty}\Gamma(\{p_n\})=\infty$. Since $1=\Gamma(E)\geqq\Gamma(\sum_{n=1}^{\infty}\{p_n\})$, a contradiction is apparent.

## Bibliography

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