A REMARK ON ALGEBRAS OF MATRICES

WINSTON M. SCOTT

1. **Introduction.** Let $\mathfrak{A}$ denote a matrix algebra, with unit element, over an algebraically closed field K. We shall assume that $\mathfrak{A}$ is in reduced form, that is, that $\mathfrak{A}$ is exhibited with only zeros above the main diagonal, with irreducible constituents of $\mathfrak{A}$ in the main diagonal, and that $\mathfrak{A}$ is expressible as the direct sum of its radical and a semisimple subalgebra which latter has nonzero components only in the irreducible constituents of $\mathfrak{A}$:

$$\mathfrak{A} = \begin{pmatrix} \mathfrak{C}_{11} & \cdot & \cdots & \cdot \\ \mathfrak{C}_{21} & \mathfrak{C}_{22} & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \mathfrak{C}_{t1} & \mathfrak{C}_{t2} & \cdots & \mathfrak{C}_{tt} \end{pmatrix}, \tag{1}$$

the $\mathfrak{C}_{ii}$ denoting irreducible constituents; further $\mathfrak{A}=\mathfrak{A}^*+\mathfrak{N}$ where $\mathfrak{N}$ is the radical of $\mathfrak{A}$ and

$$\mathfrak{N} = \begin{pmatrix} 0 & \cdot & \cdots & \cdot \\ \mathfrak{C}_{21} & 0 & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \mathfrak{C}_{t1} & \mathfrak{C}_{t2} & \cdots & 0 \end{pmatrix}, \qquad \mathfrak{A}^* = \begin{pmatrix} \mathfrak{C}_{11} & \cdot & \cdots & \cdot \\ 0 & \mathfrak{C}_{22} & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & \mathfrak{C}_{tt} \end{pmatrix}. \tag{2}$$

As a part of $\mathfrak{A}$, $\mathfrak{C}_{ij}$ forms an additive group or module of matrices upon which $\mathfrak{A}$, itself considered as a module, is homomorphically mapped. We shall consider $\mathfrak{C}_{ij}$ as a matrix module with $\mathfrak{A}$ as both left and right operator system. For a matrix A of $\mathfrak{A}$, we shall use the notation $C_{ij}(A)$, $(j \leqq i,\ i=1, 2, \cdots, t)$, to denote the parts of A,

$$A = \begin{pmatrix} C_{11}(A) & \cdot & \cdots & 0 \\ C_{21}(A) & C_{22}(A) & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ C_{t1}(A) & \cdot & \cdots & C_{tt}(A) \end{pmatrix}. \tag{3}$$

Let B be any element of $\mathfrak{A}$, and let B^* be the component of B in the semisimple subalgebra $\mathfrak{A}^*$. We define B as a left and as a right operator of $C_{ij}(A)$ by the relations below, using $\circ$ to distinguish this operation from ordinary matrix multiplication

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$$(4) \qquad \begin{aligned} B \circ C_{ij}(A) &= C_{ii}(B) \cdot C_{ij}(A) = C_{ij}(B^*A), \\ C_{ij}(A) \circ B &= C_{ij}(A) \cdot C_{jj}(B) = C_{ij}(AB^*). \end{aligned}$$

We shall indicate that a matrix module has $\mathfrak{A}$ as both right and left operator system by calling the module an $(\mathfrak{A}, \mathfrak{A})$ module. Under (4), $\mathfrak{C}_{ij}$ is a simple $(\mathfrak{A}, \mathfrak{A})$ module. Moreover, each $\mathfrak{C}_{ij}$ is either a 0-part or there exist elements in $\mathfrak{A}$ such that the corresponding C_{ij} have any arbitrary components from K. We shall call the $\mathfrak{C}_{ij}$ *simple parts* of $\mathfrak{A}$.[1]

2. **The basic theorem.** Professor R. Brauer, in a recent paper,[2] has proved a theorem which has a great many applications and among other things includes the Jordan-Hölder Theorem as a special case.

BASIC THEOREM. *Let G and H be two groups, with finite composition lengths, for both of which a given set θ is the operator set. Let*

$$G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_r = (1)$$

be a composition series of G, and

$$H = H_0 \supset H_1 \supset H_2 \supset \cdots \supset H_s = (1)$$

be a composition series of H. If θ is a homomorphism which maps H upon a normal subgroup H^* of G, $H^* \subseteq G$, then one can choose complete residue systems P_ρ of $G_{\rho-1}$ (mod G_ρ) *and Q_σ of $H_{\sigma-1}$* (mod H_σ), $(\rho=1, 2, \cdots, r; \sigma=1, 2, \cdots, s)$, *such that*

(a) *either θ maps Q_σ on a P_ρ in a $(1-1)$ manner and $G_{\rho-1}/G_\rho = H_{\sigma-1}/H_\sigma$, or θ maps Q_σ on* 1, *and*

(b) *each P_ρ is the image of at most one Q_σ,.*

Here, in our application of the basic theorem, H will be mapped on the whole group G and, consequently, statement (b) of the theorem may be sharpened to "*Each P_ρ is the image of exactly one Q_σ.*"

3. **An application of the basic theorem.** We shall first prove:

THEOREM 1. *A composition series*

$$\mathfrak{A} = \mathfrak{A}_0 \supset \mathfrak{A}_1 \supset \mathfrak{A}_2 \supset \cdots \supset \mathfrak{A}_m$$

of a matrix algebra $\mathfrak{A}$ considered as an $(\mathfrak{A}, \mathfrak{A})$ module is also a composition series of $\mathfrak{A}$ considered as an $(\mathfrak{A}^*, \mathfrak{A}^*)$ module where $\mathfrak{A}$ is considered as the direct sum of $\mathfrak{A}^*$, a semisimple subalgebra, and $\mathfrak{N}$, the radical of $\mathfrak{A}$.

[1] For a further study of simple parts, see W. M. Scott, *On matrix algebras over an algebraically closed field*, Ann. of Math. vol. 43 (1942) pp. 147–161.

[2] R. Brauer, *On sets of matrices with coefficients in a division ring*, Trans. Amer. Math. Soc. vol. 49 (1941) pp. 502–548.

Proof. In $\mathfrak{A}_{i-1}/\mathfrak{A}_i$ we consider all residue classes $\langle A^{(i-1)}\rangle$. Select the classes different from $\langle 0\rangle$ which have representatives belonging to[3] a highest power of the radical $\mathfrak{N}$, say $\mathfrak{N}^\rho$. These together with the $\langle 0\rangle$ class form an admissible subgroup with elements of $\mathfrak{A}$ as operators on both left and right (that is, as $(\mathfrak{A}, \mathfrak{A})$ operators), for if we have $\langle A^{(i-1)}\rangle$ such that $A^{(i-1)}$ belongs to $\mathfrak{N}^\rho$, then

$$A\langle A^{(i-1)}\rangle B = \langle AA^{(i-1)}B\rangle$$

has the representative $AA^{(i-1)}B$ which is contained in $\mathfrak{N}^\rho$. Then this admissible subgroup must coincide with the whole factor group $\mathfrak{A}_{i-1}/\mathfrak{A}_i$ since $\mathfrak{A}_{i-1}/\mathfrak{A}_i$ is simple. We can conclude, then, that every class in $\mathfrak{A}_{i-1}/\mathfrak{A}_i$ has a representative which belongs to $\mathfrak{N}^\rho$.

Now let $\langle A^{(i-1)}\rangle$ be a class with $A^{(i-1)}$ belonging to $\mathfrak{N}^\rho$. We have that for $N\in\mathfrak{N}$

$$N\langle A^{(i-1)}\rangle = \langle NA^{(i-1)}\rangle$$

has a representative in $\mathfrak{N}^{\rho+1}$. But we have selected all classes different from $\langle 0\rangle$ belonging to a highest power which was $\mathfrak{N}^\rho$. Then the class $\langle NA^{(i-1)}\rangle$ must be the 0-class. Therefore, we have that for $A\in\mathfrak{A}$, $A=A^*+N$, $A^*\in\mathfrak{A}^*$, and $N\in\mathfrak{N}$,

$$\begin{aligned} A\langle A^{(i-1)}\rangle &= \langle A\cdot A^{(i-1)}\rangle \\ &= \langle A^*\cdot A^{(i-1)} + N\cdot A^{(i-1)}\rangle \\ &= \langle A^*\cdot A^{(i-1)}\rangle \\ &= A^*\langle A^{(i-1)}\rangle, \end{aligned}$$

and our theorem is proved.

Now considering the simple part $\mathfrak{C}_{ij}$ as an $(\mathfrak{A}^*, \mathfrak{A}^*)$ module (as we may, since $\mathfrak{A}^*\subset\mathfrak{A}$), we have that $A\rightarrow C_{ij}(A)$ is an $(\mathfrak{A}^*, \mathfrak{A}^*)$ homomorphism so that by our basic theorem $\mathfrak{C}_{ij}$ is $(\mathfrak{A}^*, \mathfrak{A}^*)$ isomorphic to a composition factor group of $\mathfrak{A}$. But the operators of the system $\mathfrak{A}$ on $\mathfrak{C}_{ij}$ and on the factor group are equivalent by (4) and Theorem 1 to operators of the system $\mathfrak{A}^*$. From this we have that $\mathfrak{C}_{ij}$ is also $(\mathfrak{A}, \mathfrak{A})$ isomorphic to the composition factor group.

Thus we have, by the basic theorem, a proof of the following theorem.

Theorem 2.[4] *A nonzero simple part $\mathfrak{C}_{ij}$ of $\mathfrak{A}$ is $(\mathfrak{A}, \mathfrak{A})$ isomorphic to a composition factor group of $\mathfrak{A}$ itself considered as an $(\mathfrak{A}, \mathfrak{A})$ module.*

Washington, D. C.

[3] For a definition of *belonging to* as used here, see C. Nesbitt, *On the regular representations of algebras*, Ann. of Math. vol. 39 (1938) pp. 634–658.

[4] For a direct proof of this theorem, see W. M. Scott, op. cit.