$$x_0^2 = u^2, \qquad x_n^2 = u^2, \qquad\qquad 0 \leqq u^2 \leqq 1,$$

$$x_0^3 = 0, \qquad x_n^3 = \frac{1}{n} + \frac{\sin n^4 u^1}{n^3}.$$

Then we have

$$\begin{aligned}\liminf_n \iint_{B_n} f(x_n, X_n) du &= \liminf_n \int_0^{\pi}\int_0^1 \left| 1 - \cos^2 n^4 u^1 \right| du^2 du^1 \\ &= \frac{\pi}{2} < \pi = \int_0^{\pi}\int_0^1 du^2 du^1 \\ &= \iint_{B_0} f(x_0, X_0) du.\end{aligned}$$

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A NON-INVOLUTORIAL SPACE TRANSFORMATION ASSOCIATED WITH A $Q_{1,n}$ CONGRUENCE

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1. **Introduction.** The involutorial transformation associated with the congruence of lines meeting a curve of order m and an $(m-1)$-fold secant has been studied by DePaolis,[1] and Vogt[2] has studied the non-involutorial transformations for a linear congruence and bundle of lines. Cunningham[3] has recently studied some non-involutorial transformations associated with a $Q_{1,2}$ congruence. In the present paper a non-involutorial transformation associated with the congruence of lines on a plane curve of order n having an $(n-1)$-point and a secant through that point is considered. The bundle of lines through the multiple point is not considered as belonging to the congruence. The tangents to the curve at the point are considered to be distinct.

Received by the editors November 26, 1941.

[1] DePaolis, *Alcuni particolari transformazioni involutori dello spazio*, Rendiconti dell'Academia dei Lincei, Rome, (4), vol. 1 (1885), pp. 735–742, 745–758.

[2] Vogt, *Zentrale und windschiefe Raum-Verwandtschaften*, Jahresbericht der Schlesischen Gesellschaft für Vaterlandische Kultur, class 84, 1906, pp. 8–16.

[3] Cunningham, *Non-involutorial space transformations associated with a $Q_{1,2}$ congruence*, this Bulletin, vol. 47 (1941), pp. 309–312.

Given the plane n-ic r, a line s meeting r at an $(n-1)$-point A, and two projective pencils of surfaces $|F_m|: s^{m-1}g_{2m-1}$ and $|F'_{m'}|: s^{m'-1}g_{2m'-1}$. Through a generic point $P(y)$ there passes a single F of $|F|$. The unique line t through $P(y)$, s, r meets the associated F' of $|F'|$ in one residual point $P'(x)$ the image of $P(y)$ under the transformation thus defined. The residual base curves of $|F|$ and $|F'|$, other than s, have been denoted by g and g', respectively. Through a point $O_{g'}$ on g' there is a unique line t' of the congruence, this line lying upon one surface of $|F'|$. The associated surface of $|F|$ meets t' in a point $\overline{P}$ which generates a curve $\bar{g}$. Similarly, beginning with a point O_g on g, a point $\overline{P}'$ generating a curve $\bar{g}'$ is found. It will be shown that r, s, g, g', $\bar{g}$, $\bar{g}'$ are fundamental curves of the transformation, and that the point A is a fundamental point of the second kind.

2. **Equations of the transformation.** Let us take the equations of r and s, respectively, as

$$x_3[cx_1x_2] - [dx_1x_2] = 0, \qquad x_4 = 0, \qquad x_1 = x_2 = 0 \tag{1}$$

where

$$[cx_1x_2] = \sum_{i=0}^{n-1} c_{i,n-i-1}\overset{i}{x_1}\overset{n-i-1}{x_2}, \quad [dx_1x_2] = \sum_{j=0}^{n} d_{j,n-j}\overset{j}{x_1}\overset{n-j}{x_2}, \tag{2}$$

and the pencils of surfaces as

$$|F_m| \equiv U - uV = 0, \qquad |F'_{m'}| \equiv U' - uV' = 0 \tag{3}$$

where

$$U = (ax)\{ex_1x_2\} - (\alpha x)\{fx_1x_2\}, \qquad V = (bx)\{gx_1x_2\} - (\beta x)\{hx_1x_2\},$$

$$U' = (a'x)\{e'x_1x_2\} - (\alpha' x)\{f'x_1x_2\}, \quad V' = (b'x)\{g'x_1x_2\} - (\beta' x)\{h'x_1x_2\};$$

$$\{ex_1x_2\} = \sum_{p=0}^{m-1} e_{p,m-p-1}\overset{p}{x_1}\overset{m-p-1}{x_2}, \quad \{e'x_1x_2\} = \sum_{p=0}^{m'-1} e'_{p,m'-p-1}\overset{p}{x_1}\overset{m'-p-1}{x_2} \tag{4}$$

$$(ax) = a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4,$$

and so on.

Through a generic point $P(y)$ there is an F of $|F|$ with parameter $u = U(y)/V(y)$, and to this corresponds the F' of $|F'|$ having equation

$$U'(x)V - V'(x)U = 0. \tag{5}$$

The unique transversal t through P, r, s meets (5) in the point P' having coordinates

$$(6)\qquad T^{-1}_{m+m'+n}:\left|\begin{aligned}\sigma x_1 &= Ry_1 + Ky_1[cy_1y_2] = y_1S,\\ \sigma x_2 &= Ry_2 + Ky_2[cy_1y_2] = y_2S,\\ \sigma x_3 &= Ry_3 + K[dy_1y_2],\\ \sigma x_4 &= Ry_4,\end{aligned}\right.$$

where

$$(7)\qquad \begin{aligned} K_{m+m'} &= U'V - V'U, \qquad R_{m+m'+n-1} = UW' - VZ',\\ W'_{m'+n-1} &= (b'z)\{g'y_1y_2\} - (\beta'z)\{h'y_1y_2\},\\ Z'_{m'+n-1} &= (a'z)\{e'y_1y_2\} - (\alpha'z)\{f'y_1y_2\},\\ (a'z) &= (a_1'y_1 - a_2'y_2)[cy_1y_2] + a_3'[dy_1y_2],\end{aligned}$$

and so on. Equations (6) are those of the inverse transformation.

In a similar manner the equations of the direct transformation are found to be

$$(8)\qquad T_{m+m'+n}:\left|\begin{aligned}\tau y_1 &= R'x_1 + K'x_1[cx_1x_2] = x_1S',\\ \tau y_2 &= R'x_2 + K'x_2[cx_1x_2] = x_2S',\\ \tau y_3 &= R'x_3 + K'[dx_1x_2],\\ \tau y_4 &= R'x_4,\end{aligned}\right.$$

where

$$(9)\qquad \begin{aligned} K'_{m+m'} &= UV' - U'V = -K, \quad R'_{m+m'+n-1} = U'W - V'Z,\\ W_{m+n-1} &= (bz)\{gx_1x_2\} - (\beta z)\{hx_1x_2\},\\ Z_{m+n-1} &= (az)\{ex_1x_2\} - (\alpha z)\{fx_1x_2\}.\end{aligned}$$

3. **Images of fundamental curves and elements.** The transformations T^{-1} and T applied to an F' and F of $|F'|$ and $|F|$, respectively, give $U' \sim (T^{-1})US^{m'-1}G$, $U \sim (T)U'S'^{m-1}G'$ where

$$(10)\qquad \begin{aligned} G_{2m'+n-1} &= W'U' - V'Z',\\ G'_{2m+n-1} &= WU - VZ,\\ S_{m+m'+n-1} &= UN' - VM',\\ S'_{m+m'+n-1} &= U'N - V'M,\\ M'_{m'+n-1} &= (a'w)\{e'y_1y_2\} - (\alpha'w)\{f'y_1y_2\},\\ M_{m+n-1} &= (aw)\{ex_1x_2\} - (\alpha w)\{fx_1x_2\},\\ N'_{m'+n-1} &= (b'w)\{g'y_1y_2\} - (\beta'w)\{h'y_1y_2\},\\ N_{m+n-1} &= (bw)\{gx_1x_2\} - (\beta w)\{hx_1x_2\},\\ (a'w) &= a_3'[dy_1y_2] - (a_3'y_3 + a_4'y_4)[cy_1y_2].\end{aligned}$$

Here U and U' are the corresponding surfaces of $|F|$ and $|F'|$ and $g'\sim(T^{-1})G$, $s'\sim(T^{-1})S$, $g\sim(T)G'$, $s\sim(T)S'$.

Similarly

$$
(11)\quad
\begin{aligned}
&K'\sim(T^{-1})\ K'S^{m+m'-2}GG', && K\sim(T)\ KS'^{m+m'-2}GG',\\
&K'\sim(T)\ K'S'^{m+m'-2}GG', && K\sim(T^{-1})\ KS^{m+m'-2}GG',\\
&G'\sim(T^{-1})\ RS^{2m+n-2}G', && G\sim(T)\ R'S'^{2m'+n-2}G,\\
&G'\sim(T)\ R'S'^{2m+n-2}G', && G\sim(T^{-1})\ RS^{2m'+n-2}G,\\
&R'\sim(T^{-1})\ S^{m+m'+n-2}GG', && R\sim(T)\ S'^{m+m'+n-2}GG',\\
&R'\sim(T)\ S'^{m+m'+n-2}[R'^2+K'(W'Z-WZ')], &&\\
&R\sim(T^{-1})\ S^{m+m'+n-2}[R^2+K(W'Z-WZ')], &&\\
&S'\sim(T^{-1})\ RS^{m+m'+n-3}GG', && S\sim(T)\ R'S'^{m+m'+n-3}GG',\\
&S'\sim(T)\ S'^{m+m'+n-3}\{R'^2S'+R'GG'+S'[GG'+K'(W'Z-WZ')]\}, &&\\
&S\sim(T^{-1})\ S^{m+m'+n-3}\{R^2S+RGG'+S[GG'+K(W'Z-WZ')]\}. &&
\end{aligned}
$$

Through a point O_r on r there is a pencil of transversals through s. O_r determines an F' and the associated F cuts the pencil in s and a line l. The line l generates the ruled surface R, the image of r.

From a point O_s on s there is an n-ic cone of transversals to r, to each line of which corresponds one F of $|F|$ cutting that line in one point. The locus of all such points is a curve k which generates the surface S, the image of s. The order of k, determined by the intersection of S and a homaloidal surface, is $m+m'+n-2$.

Through a point $O_{g'}$ on g' there is a unique line t of the congruence, but every F of $|F|$ passes through $O_{g'}$, hence $O_{g'}\sim(T^{-1})t$. The ruled surface G generated by t is the image of g' under T. Furthermore, every point P' of the line determines the same F' and t meets the associated F in one point $\overline{P}$ so that $\overline{P}\sim(T)t$. The locus of points $\overline{P}$ is the curve $\bar{g}$ and $\bar{g}\sim(T)G$. The order of $\bar{g}$, determined by the intersection of two homaloidal surfaces, is $m+3m'+2n-3$. In a similar manner we find a curve $\bar{g}'$, of order $3m+m'+2n-3$, such that

$$\bar{g}'\sim(T^{-1})G'.$$

The multiple point A is a fundamental point of the second kind and has as an image $n-1$ lines $c_{1,i}$, $i=1,\cdots,n-1$, other than s lying one in each of the $n-1$ planes determined by s and the tangent lines to r at A.

We can now write the following correspondences:

$$
(12)\qquad
\begin{aligned}
&r \sim (T)\, R': s^{m+m'+n-2} g' \bar{g}' c'_{1,1} \cdots c'_{1,n-1},\\
&r \sim (T^{-1})\, R: s^{m+m'+n-2} g \bar{g} c_{1,1} \cdots c_{1,n-1},\\
&s \sim (T)\, S': r s^{m+m'+n-3} g' \bar{g}' c'_{1,1} \cdots c'_{1,n-1},\\
&s \sim (T^{-1})\, S: r s^{m+m'+n-3} g \bar{g} c_{1,1} \cdots c_{1,n-1},\\
&g \sim (T)\, G': r s^{2m+n-2} g \bar{g}', \qquad g' \sim (T^{-1})\, G: r s^{2m'+n-2} g' \bar{g},\\
&\bar{g}' \sim (T)\, G': r s^{2m+n-2} g \bar{g}', \qquad \bar{g} \sim (T)\, G: r s^{2m'+n-2} g' \bar{g}.
\end{aligned}
$$

4. **Invariant and homaloidal surfaces.** The eliminant of the parameter from $|F|$ and $|F'|$ is the pointwise invariant surface K. The plane $x_4=0$ and the planes determined by s and the tangent lines to r at A are also invariant, but not pointwise invariant.

Generic planes subjected to the transformations give

$$
\begin{aligned}
\pi' &\equiv (A'x) \sim (T^{-1})\, R(A'y) + K(A'z) = \phi_{m+m'+n},\\
\pi &\equiv (Ay) \sim (T)\, R'(Ax) + K'(Az) = \phi'_{m+m'+n},
\end{aligned}
$$

where the ϕ's are homaloidal surfaces of the transformations. Further,

$$
\phi \sim (T)(A'x)R'S'^{\,m+m'+n-2}GG', \qquad \phi' \sim (T^{-1})\,(Ay)RS^{m+m'+n-2}GG',
$$

hence the homaloidal webs are

$$
\infty^3 \,|\, \phi \,|\; : r s^{m+m'+n-2} g \bar{g}, \qquad \infty^3 \,|\, \phi' \,|\; : r s^{m+m'+n-2} g' \bar{g}'.
$$

The intersection of two homaloidal surfaces gives the homaloidal net

$$
\begin{aligned}
H' &\equiv [\phi'\phi'] : r s^{m^2+m'^2+n^2+2mm'+2m'n-4m-4m'-4n+4} g' \bar{g}' k_{m+m'+n}\\
H &\equiv [\phi\phi] : r s^{m^2+m'^2+n^2+2mm'+2mn+2m'n-4m-4m'-4n+4} g \bar{g} k_{m+m'+n}.
\end{aligned}
$$

We now write the additional correspondences:

$$
(13)\qquad
\begin{aligned}
&\pi \sim (T)\, \phi' : r s^{m+m'+n-2} g' \bar{g}', \qquad \pi' \sim (T^{-1})\, \phi : r s^{m+m'+n-2} g \bar{g},\\
&K \sim (T)\, K' : s^{m+m'-2} g g' \bar{g} \bar{g}', \qquad K' \sim (T^{-1})\, K : s^{m+m'-2} g g' \bar{g} \bar{g}'.
\end{aligned}
$$

The jacobian of the transformation is $J=RGG'S$.

5. **Tangency along** s. The projectivity $y_1=x_1$, $y_2=x_2$, $y_3=kx_3$, $y_4=x_3+x_4$ is applied to the fundamental surfaces of the transformation and an examination of the coefficients of the highest powers of x_3 shows K and S to have

$$
\begin{aligned}
D_{m+m'-2} \equiv\; & [(a_3'k + a_4')\{e'x_1x_2\} - (\alpha_3'k + \alpha_4')\{f'x_1x_2\}]\\
&\quad \cdot [(b_3k + b_4)\{gx_1x_2\} - (\beta_3k + \beta_4)\{hx_1x_2\}]\\
&- [(b_3'k + b_4')\{g'x_1x_2\} - (\beta_3'k + \beta_4')\{h'x_1x_2\}]\\
&\quad \cdot [(a_3k + a_4)\{ex_1x_2\} - (\alpha_3k + \alpha_4)\{fx_1x_2\}] = 0
\end{aligned}
$$

as common tangent surface along s. The same surface is tangent to K' and S' also along s.

6. **Intersection table.** Referring to (12), (13) and §5 we can now write the following intersection table:

$$[R'S']: s^{m^2+m'^2+n^2+2mm'+2mn+2m'n-5m-5m'-5n+6} g'\bar{g}' c_{1,1} \cdots c_{1,n-1}$$
$$[R'G']: s^{2m^2+n^2+2mm'+3mn+m'n-6m-4n-2m'+4} \bar{g}'$$
$$[R'\phi']: s^{m^2+m'^2+n^2+2mm'+2mn+2m'n-4m-4m'-4n+4} g'\bar{g}' l_{1,1} \cdots l_{1,n}$$
$$[R'K']: s^{m^2+m'^2+2mm'+mn+m'n-4m-4m'-2n+4} g'\bar{g}'$$
$$[S'G']: rs^{2m^2+n^2+2mm'+3mn+m'n-8m-2m'-5n+6} \bar{g}' k_{1,1} \cdots k_{1,2m-2}$$
$$[S'\phi']: rs^{m^2+m'^2+n^2+2mm'+2mn+2m'n-5m-5m'-5n+6} g'\bar{g}' k_{m+m'+n-2}$$
$$[S'K']: s^{m^2+m'^2+2mm'+mn+m'n-5m-5m'-2n+6+(m+m'-2)d} g'\bar{g}'$$
$$[G'\phi']: rs^{2m^2+n^2+2mm'+3mn+m'n-6m-4n-2m'+4} \bar{g}' j_{1,1} \cdots j_{1,2m-1}$$
$$[G'K']: s^{2m^2+mn+2mm'+m'n-6m-2m'-2n+4} g\bar{g}'$$
$$[\phi'K']: S^{m^2+m'^2+2mm'+mn+m'n-4m-4m'-2n+4} g'\bar{g}' k_{m+m'}.$$

7. **The T_2 in a plane through s.** A plane $\pi \equiv x_1 = \sigma x_2$ cuts the surfaces of $|F_m|$ and $|F'_{m'}|$ in residual pencils of lines

$$|l| \equiv u - \mu v = 0, \qquad |l'| \equiv u' - \mu v' = 0$$

where $u = (ax)\{e\sigma\} - (\alpha x)\{f\sigma\}$, $u' = (a'x)\{e'\sigma\} - (\alpha' x)\{f'\sigma\}$, $v = (bx)\{g\sigma\} - (\beta x)\{h\sigma\}$, $v' = (b'x)\{g'\sigma\} - (\beta' x)\{h'\sigma\}$, $(ax) = (a_1\sigma + a_2)x_2 + a_3x_3 + a_4x_4$, $\{e\sigma\} = \sum_{p=0}^{m-1} e_{p,m-p-1}\sigma^p$, $\{e'\sigma\} = \sum_{p=0}^{m'-1} e'_{p,m'-p-1}\sigma^p$, and so on. The n-ic r intersects π in one residual point P: $(\sigma[c\sigma],\ [c\sigma],\ [d\sigma],\ 0)$ where $[c\sigma] = \sum_{i=0}^{n-1} c_{i,n-i-1}\sigma^i$, $[d\sigma] = \sum_{j=0}^{n} d_{j,n-j}\sigma^j$, and the vertices of $|l|$ and $|l'|$ are designated as Γ and Γ'.

Through a generic point $P(y)$ of π passes one l of $|l|$ having parameter $\mu = u(y)/v(y)$ and to this corresponds the line $u'v - uv' = 0$ which is met by l in a point $P'(x)$, image of P under T. The T_2^{-1} in π is thus

$$(14) \qquad x_2 = \rho y_2 + \kappa[c\sigma], \qquad x_3 = \rho y_3 + \kappa[d\sigma], \qquad x_4 = \rho y_4,$$

where $\rho_1 = uw' - vz'$, $\kappa_2 = u'v - v'u$, $z' = (a'q)\{e'\sigma\} - (\alpha' q)\{f'\sigma\}$, $w' = (b'q)\{g'\sigma\} - (\beta' q)\{h'\sigma\}$, $(a'q) = (a_1'\sigma + a_2')[c\sigma] + a_3'[d\sigma]$, and so on.

The direct transformation T is

$$(15) \qquad y_2 = \rho' x_2 + \kappa'[c\sigma], \qquad y_3 = \rho' x + \kappa'[d\sigma], \qquad y_4 = \rho' x_4$$

where $\rho_1' = u'w - v'z$, $\kappa' = -\kappa$.

The conic $\kappa:\Gamma\bar{\Gamma}\Gamma'\bar{\Gamma}'$ is pointwise invariant under the transformation.

Through the point Γ there is one line γ' through P and every line of $|l'|$ corresponds to Γ, hence γ' is the image of Γ under T. Moreover, every point of γ' determines the same line of $|l'|$ which intersects γ' in a point $\bar{\Gamma}'$ whose image under T^{-1} is also γ'. Similarly, a γ of $|l|$ is the image of Γ' and $\bar{\Gamma}$ under T^{-1} and T, respectively.

The point P is the vertex of a pencil of transversals. Moreover, through P there passes one line of $|l|$, hence there corresponds one line of $|l'|$. This line is met in every point by a line of the pencil, hence is the image of P under T.

Thus the points P, Γ, $\bar{\Gamma}$ and P, Γ', $\bar{\Gamma}'$ are fundamental under T_2 and T_2^{-1}, respectively, so that we have $P\sim(T)\ \rho':\Gamma'\bar{\Gamma}'$, $\Gamma\sim(T)\,\gamma':P\Gamma\bar{\Gamma}'$, $\bar{\Gamma}\sim(T)\ \gamma:P\Gamma'\bar{\Gamma}$ and $P\sim(T)\ \rho:\Gamma\bar{\Gamma}$, $\bar{\Gamma}'\sim(T^{-1})\ \gamma:P\Gamma'\bar{\Gamma}$, $\bar{\Gamma}'\sim(T^{-1})$ $\gamma':P\Gamma\bar{\Gamma}'$.

The homaloidal nets of T and T^{-1} are

$$\infty^2|f_2'|:P\Gamma'\bar{\Gamma}', \qquad \infty^2|f_2|:P\Gamma\bar{\Gamma}$$

while the jacobian of T_2 is $j=\rho'\gamma\gamma'$ and of T_2^{-1} is $j'=\rho\gamma\gamma'$.

As the plane π generates the pencil on s the T generates the space $T_{m+m'+n}$ whose equations may be obtained from (14) and (15) by replacing u, u', v, v', w, w', z, z' and σ by U, U', V, V', W, W', Z, Z' and x_1/x_2, respectively.

Since the point P is the section of r by π, r may be represented by $x_1=\sigma[c\sigma]$, $x_2=[c\sigma]$, $x_3=[d\sigma]$, $x_4=0$.

8. r **having** $(n-1)$**-point with coincident tangents.** In case k of the tangents to r at A are coincident the transformation will be identical with the above except for the image of A, the correspondences involved then being

$$r\sim(T)\ R':s^{m+m'+n-2}g'\bar{g}'c_{1,1}'^{k},\ c_{1,k+1}'\cdots c_{1,n-1}',$$
$$s\sim(T)\ S':rs^{m+m'+n-3}g'\bar{g}'c_{1,1}',\ c_{1,k+1}'\cdots c_{1,n-1}',$$

and the intersection

$$[R'S']:s^{m^2+m'^2+n^2+2mm'+2mn+2m'n-5m-5m'-5n+6}g'\bar{g}'c_{1,1}'^{k},\ c_{1,k+1}'\cdots c_{1,n-1}'.$$

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