INDECOMPOSABLE CONNEXES[1]

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DEFINITION. *A connected set M is an indecomposable connexe if and only if, for every two connected subsets H and K of M such that $M=H+K$, either H and M or K and M have the same closure.*[2]

Any connected subset N of an indecomposable continuum W, which is dense in W, such as any set of composants of W or W itself, is an indecomposable connexe, as is also a widely connected set.[3]

EXAMPLE A.[4] Let, in a euclidean plane, U be the points of a square, Q, plus its interior. Let U_i ($i=1, 2, 3, \cdots$) be a set of mutually exclusive arcs each contained in U and having one and only one point, an end point, common with Q. Let the U_i's be taken so that every plane region of U is joined to every linear region of Q by at least one U_i. Let $M=U-(U_1+U_2+\cdots)$. Then M is connected[5] and such that, if H and K are connected and their sum is M, either H and M or K and M have the same closure. Hence M is an indecomposable connexe.

EXAMPLE B. Let, in a euclidean plane, U be the points of a triangle plus its interior, one vertex of which is the point a. Let U_i ($i=1, 2, 3, \cdots$) be a set of arcs, mutually exclusive, except for having the common end point a, and whose sum is dense in U. Let further the U_i's be taken so that each two plane regions of U are joined by at least one U_i. Let $M=U-(U_1+U_2+\cdots)$. It can be shown without difficulty that M is an indecomposable connexe.

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[2] See S. Eilenberg, *Topology du plan*, Fundamenta Mathematicae, vol. 26, p. 81, for a definition of an indecomposable connected space. This definition is seen to be equivalent to the above for the types of spaces considered in these two papers.

[3] For definition and example see P. M. Swingle, *Two types of connected sets*, this Bulletin, vol. 37 (1931), pp. 254–258.

[4] E. W. Miller communicated this interesting example to me by letter in 1937 calling attention to its relation to a widely connected set. The method of construction is somewhat similar to the well known boring process used to obtain a plane indecomposable continuum. See K. Yoneyama, *Theory of continuous sets of points*, Tôhoku Mathematical Journal, vol. 12 (1917), p. 60. That either H and M or K and M have the same closure is seen above by supposing that neither H nor K is dense in M, from which it readily follows that H and K can each have at most one point common with Q itself.

[5] E. W. Miller, *Some theorems on continua*, this Bulletin, vol. 46 (1940), p. 153, Theorem 3.

It is proposed to give here a generalization of some of the well known theorems on indecomposable continua[6] by means of indecomposable connexes and the following definitions. The imbedding space will be one satisfying R. L. Moore's Axioms 0 and 1.[7]

DEFINITIONS. *A connected subset K of a connected set M will be called a proper connexe subclosure of M if and only if M and K do not have the same closure. A connected set M is an irreducible connexe closure between two points a and b if and only if M contains $a+b$ and there does not exist a proper connexe subclosure of M containing $a+b$. A connected set M is an irreducible joining connexe closure between a and b if and only if there exists a subset N of M such that both N and $N+a+b$ are connected and, for all such N's, M and N have the same closure.*

Both a continuum and a connected set, irreducible between two points, are irreducible connexe closures between these two points. Also a widely connected set is an irreducible connexe closure between any two of its points. It is seen readily that *if M is an irreducible connexe closure* between a and b, then *M is an irreducible joining connexe closure between a and b.*

EXAMPLE C. In a euclidean plane let B be a biconnected set with dispersion point a and containing the point b distinct from a. Let W be an arc-wise connected set such that (a) if x and y are any two points of W then $W+a$ contains arcs ax and ay such that one of these contains the other, (b) for each x there exists but one arc ax, (c) the closure of $W+a-ax$ contains B, and (d) the product of ax and the closure of B is a. Then $M=W+B-a-b$ is an irreducible joining connexe closure from a to b, since each connected subset N of M, such that $N+a+b$ is connected, contains W. However $M+a+b$ is not an irreducible connexe closure from a to b, since $M+a+b$ contains B, which contains $a+b$, and B and M do not have the same closure.

DEFINITIONS. *A connected subset K of a connected set M is a connexe of condensation of M if and only if every point of K is a limit point of $M-K$. If M is connected a composant of $M+$ is a set of points K_p, which consists of a point p, of the closure of M but not necessarily of M, and of all points x of M such that there exists a proper connexe subclosure containing $p+x$ and contained in M excepting perhaps for p.*

[6] Brouwer, *Zur Analysis situs*, Mathematische Annalen, vol. 68 (1910), p. 426, gave the first example and definition of indecomposable continuum. For theorems on these sets see Z. Janiszewski and C. Kuratowski, *Sur les continus indécomposables*, Fundamenta Mathematicae, vol. 1, p. 215.

[7] *Foundations of Point Set Theory*, American Mathematical Society Colloquium Publications, vol. 13, 1932.

And K_p is a composant of M if and only if p is also contained in M, i.e., if K_p is a component of $M+$ but is contained entirely in M.

In a widely connected set M each composant of M consists of but one point, and each composant of $M+$ may consist of but one point. Hence it is not true that if K is such a composant of M every point of M is a limit point of K, which is however a useful theorem on indecomposable continua.[8]

THEOREM 107′. *Every composant of $M+$, where M is connected and its closure is compact, is the sum of a countable number of proper connexe subclosures, each contained in M except perhaps for one point of the closure of M.*

PROOF. Let a be a point of the closure of M and let K denote the composant of $M+$ consisting of a and all points x of M such that $M+a$ is not an irreducible connexe closure from a to x. Then there exists[9] a countable set G of domains such that if q is any point of the closure of M and D is any domain containing q there exists a domain of G, containing q, and contained wholly in D. For each domain R of G, which does not contain a, let M_R denote the maximal connected subset, containing a, of $(M+a)\cdot(S-R)$, S being the imbedding space. Let H denote the collection of all sets M_R and let T denote the sum of all these proper connexe subclosures of $M+a$ which are elements of H. The set H is countable. If q is a point of $M-T$ then $M+a$ is an irreducible connexe closure from a to q. For if there exists a proper connexe subclosure N of $M+a$, containing $a+q$, there exists a domain g of G such that the product of the closures of g and N is vacuous, where g contains a point of M. Thus N would have been contained in an M_R above and so N, and thus q, would be contained in T. Therefore K is T. Hence K is the sum of a countable number of proper connexe subclosures as the theorem states.

COROLLARY 107′. *If M is connected and its closure is compact, then every composant of M is the sum of a countable number of proper connexe subclosures of M.*

LEMMA A. *If M is an indecomposable connexe and N is a proper connexe subclosure of M, then $M-M\cdot\overline{N}$ is connected.*[10]

[8] R. L. Moore, loc. cit., Theorem 106, p. 75. Below, the theorems are numbered to correspond to similar theorems on indecomposable continua, given by Moore, pp. 75–78. It is to be noted the methods of proof are somewhat similar.

[9] R. L. Moore, loc. cit., Theorem 19, p. 14.

[10] By $\overline{N}$ is meant the closure of N.

PROOF. Suppose $M-M\cdot\overline{N}$ is the sum of the two mutually separated sets H and K. Then M is the sum of two proper connexe[11] subclosures $H+\overline{N}\cdot M$ and $K+\overline{N}\cdot M$ and so M is not indecomposable.

LEMMA A′. *If M is an indecomposable connexe and N is a proper connexe subclosure of M, then $M-N$ is connected.*

PROOF. By Lemma A $M-M\cdot\overline{N}$ is connected. Also $\overline{N}\cdot M$ is connected since N is. As M is the sum of these two sets and $M\cdot\overline{N}$ is a proper connexe subclosure, $M-M\cdot\overline{N}$ cannot be proper.

Suppose $M-N$ is the sum of the mutually separate sets U and V. But $M-N$ contains the connected set $M-M\cdot\overline{N}$ and so either U or V contains it also. Say U does. Then M and U must have the same closure. But then points of V are limit points of U which is a contradiction. Hence $M-N$ is connected.

THEOREM A. *If M is an indecomposable connexe and W a connected subset of M such that M and W have the same closure, then W is an indecomposable connexe.*

PROOF. Let $N=M-W$ and suppose $W=H+K$, H and K proper connexe subclosures of W. As N is contained in $\overline{W}=\overline{H}+\overline{K}$, let $\overline{H}\cdot N=H'$ and $\overline{K}\cdot N=K'$. Thus $H+H'$ and $K+K'$ are connected sets.[12] But as $\overline{H}$ contains the closure of $H+H'$ and $\overline{K}$ the closure of $K+K'$, $M=W+N$ is the sum of these two proper connexe subclosures and so M is not indecomposable.

COROLLARY A. *If M is an indecomposable connexe and N is both a proper connexe subclosure and a connexe of condensation of M, then $M-N$ is an indecomposable connexe.*

PROOF. By Lemma A $M-N$ is connected and by definition of connexe of condensation M and $M-N$ have the same closure. Thus the corollary follows from Theorem A.

COROLLARY A′. *If $M+f$ is an indecomposable connexe, M connected and f finite, then M is an indecomposable connexe.*

Theorem A and its corollaries treat the case where an indecomposable connexe is given and the *subtraction* of points gives an indecomposable connexe. This suggests the following *addition* problem: Let M be an indecomposable connexe and p a point of $\overline{M}-M$. *Is $M+p$ an indecomposable connexe?* This problem is left unsolved here.

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[11] R. L. Moore, loc. cit., Theorem 47, p. 33.

[12] R. L. Moore, Theorem 27, p. 17.

THEOREM 108′. *Let M be connected. Then in order that M be an indecomposable connexe it is necessary and sufficient that every proper connexe subclosure of M be a connexe of condensation of M.*

PROOF. The condition is sufficient. For suppose M is not indecomposable. Then M is the sum of two proper connexe subclosures H and K. Thus there exists a point q of H which is not a limit point of K. But K contains $M-H$. Thus q is not a limit point of $M-H$ and so H is not a connexe of condensation of M.

The condition is necessary. For suppose N is a proper connexe subclosure of M but that not every point of N is a limit point of $M-N$. By Lemma A′ $M-N$ is connected but the closures of M and $M-N$ are not the same. Hence M is the sum of two proper connexe subclosures N and $M-N$ which is a contradiction.

THEOREM 108″. *Let M be connected. Then in order that M be an indecomposable connexe it is necessary and sufficient that the closure of every proper connexe subclosure of M be a continuum of condensation of the closure of M.*

PROOF. The condition is sufficient. For suppose H and K are as in the proof above and that q is a point of $\overline{H}$ which is not a limit point of K. As $\overline{M}=\overline{H}+\overline{K}$ and $q\cdot\overline{K}=0$ q is not a limit point of $\overline{M}-\overline{H}$ contained in $\overline{K}$. Thus $\overline{H}$ is not a continuum of condensation of $\overline{M}$.

The condition is necessary. As M is indecomposable, by Lemma A, $M-M\cdot\overline{N}$ is connected, where N is a proper connexe subclosure of M. Hence M is the sum of the two connected sets $M-M\cdot\overline{N}$ and $M\cdot\overline{N}$, the latter being a proper connexe subclosure of M. Hence $M-M\cdot\overline{N}$ is not proper and so every point of $M\cdot\overline{N}$, and so of N, is a limit point of $M-M\cdot\overline{N}$. Thus every point of $\overline{N}$ is a limit point of $M-M\cdot\overline{N}=(M+\overline{N})-\overline{N}$. Therefore every point of $\overline{N}$ is a limit point of $\overline{M}-\overline{N}$ and so $\overline{N}$ is a continuum of condensation of $\overline{M}$.

Let B be a composant of an indecomposable continuum K, where $K-B$ contains an arc A. Let c and d be two points of A such that $A-c-d=A'+A''+A'''$, where A', A'', and A''' are mutually separated sets, but $A'+c+A''$ and $A''+d+A'''$ are connected. Let $M=B+A'+A''+A'''$. Then M is an indecomposable connexe. The composant of $M+$ containing c is $A'+c+A''$ and the one containing d is $A''+d+A'''$. Thus two composants of $M+$ are not necessarily mutually exclusive.

THEOREM 109′. *If M is an indecomposable connexe, whose closure is compact, then no two composants of M have a point in common.*

PROOF. For each point p of M let M_p denote the set of points x such that M is not an irreducible connexe closure from p to x. If b is a point of M_a then $M_b = M_a$. For suppose not and that x is any point of M_a and y is of M_b. Then there exist proper connexe subclosures N_{ax}, N_{by}, N_{ab} of M. Suppose $\overline{N}_{ab} + \overline{N}_{ax} = \overline{M}$. But N_{ab} and N_{ax} are continua of condensation of $\overline{M}$ by Theorem 108″. This is a contradiction.[13] Therefore $N_{ab} + N_{ax}$ is a proper connexe subclosure of M as is similarly $(N_{ab} + N_{ax}) + N_{by}$. Hence $N_{ab} + N_{ax} + N_{by}$ is contained in both M_a and in M_b and so $M_a = M_b$. Hence if two composants have a point in common they are the same composant.

Since a composant of an indecomposable continuum is itself an indecomposable connexe it is not true that an indecomposable connexe contains uncountably many composants. A composant of $M+$ however may consist of a single point. Thus we have the following theorem.

THEOREM 110′. *If M is an indecomposable connexe whose closure is compact and, for every point p of $\overline{M} - M$, $M + p$ is an indecomposable connexe, then there exist an uncountable number of composants of $M+$.*

PROOF. Suppose there exist but a countable number of composants of $M+$. Then by Theorem 107′ M is contained in a countable number of proper connexe subclosures of $\overline{M}$. Say these are the elements of the set (N). An N of (N) contains at most one point p of $\overline{M} - M$ and by hypothesis $M + p$ is an indecomposable connexe. Hence by Theorem 108″ $\overline{N}$ is a continuum of condensation of $\overline{M+p} = \overline{M}$. But $\overline{M}$ is the sum of the $\overline{N}$'s of (N), since M is the sum of the N's. As this is a contradiction[14] the theorem is true.

THEOREM 111′. *If M is connected and its closure is compact then in order that M be an indecomposable connexe it is necessary and sufficient that there exist three distinct points such that M is an irreducible joining connexe closure between any two of them.*

PROOF. The condition is sufficient. For if M is the sum of the connexes H and K, one of these has at least two of the three points as limit points and so it and M have the same closure.

The condition is necessary. For if M contains three points x, y, and z such that each of these is in a different composant, M is an irreducible connexe closure between any two of these points. Consider the case where M contains only the one composant T, containing a

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[13] R. L. Moore, loc. cit., Theorem 15, p. 11.

[14] R. L. Moore, loc. cit., Theorem 15, p. 11.

point x. Then by Corollary 107′ M is the sum of the elements of a countable class (N), each element of which is a proper connexe subclosure. Then by Theorem 108″ every $\overline{N}$ of (N) is a continuum of condensation of $\overline{M}$. But if $\overline{M}$ is the sum of the $\overline{N}$'s this is a contradiction.[15] Hence $\overline{M}-M$ contains points y and z which are not contained in any $\overline{N}$ of (N). Thus if the connected set H of T contains x and has z as a limit point, H and M have the same closure. Thus M is an irreducible joining connexe closure from x to z and similarily from x to y. Suppose M contains a proper connexe subclosure N' which has y and z as limit points. Because of the nature of H above, N' does not contain x. From the manner of constructing the sets N of (N) in Theorem 107′, using x for the point a there, it is seen that N' is contained in an N of (N) and so does not have y or z as a limit point. Therefore M is an irreducible joining connexe closure between y and z also. In case M is the sum of two composants, y and z can be taken as above and the proof completed.[16]

THEOREM 112′. *If a is a point of an indecomposable connexe M whose closure is compact and K is the set of all points x such that M is an irreducible joining connexe closure from a to x, then K is dense in M.*

PROOF. Suppose that there exists a region R, containing a point of M, such that R does not contain a point of K. Let N be a maximal connected subset of $R\cdot M$. Then by Lemma A $M-M\cdot\overline{N}$ is connected as N is a proper connexe subclosure of M. Thus $\overline{N}$ is a continuum of condensation of $\overline{M}$. Hence[17] the locally compact closed set $\overline{M}\cdot\overline{R}$ is not the sum of the closures of a countable number of composants of $M\cdot R$. Hence by Theorem 107′ $\overline{M}\cdot\overline{R}$ is not contained in the sum of the closures of the countable number of proper connexe subclosures

[15] R. L. Moore, loc. cit., Theorem 15, p. 11.

[16] The question arises whether the condition in Theorem 111′ might be changed to "there exist three points x, y, and z such that $M+x+y+z$ is an irreducible connexe closure between any two of these." That three points might be taken so that $M+y$, say, is not an irreducible connexe closure between y and some point of M is seen by the following example. Let interior to the square Q, of Example A above, (V) be the set of straight line intervals joining a Cantor ternary set, on a line t, to a point y not on t. Take the U_i's as in Example A, except that no U_i has a point common with a V of (V). Let $B+y$ be a biconnected subset of the sum of the elements of (V), B being totally disconnected. Let $M=U-(U_1+U_2+\cdots)-$(points of the elements of $(V))+B$. Then M is indecomposable but $M+y$ is not an irreducible connexe closure between y and a point of B. See Example C above. Whether M could be taken so that each point of $\overline{M}-M$ is as y and $M+y$ is not an irreducible connexe closure between any two points is a question.

[17] R. L. Moore, loc. cit., Theorem 15, p. 11.

of the composant of M which contains a. Therefore by a proof similar to that of Theorem 111′ M is an irreducible joining connexe closure between a and some point of $(\overline{M}-M)\cdot R$. Thus K is dense in M.

If T is the sum of a countable number of proper connexe subclosures of an indecomposable connexe M, since M may be a composant of an indecomposable continuum, it is readily seen that $M-T$ may be disconnected. However, by repeated use of Lemma A, Theorem 108′, and Theorem A, the following theorem is seen to be true.

THEOREM 113′. *If T is the sum of a finite number of mutually exclusive proper connexe subclosures of an indecomposable connexe M, then $M-T$ is a non-vacuous indecomposable connexe.*

The two following theorems are proven in a manner similar to that used for the corresponding theorems on continua.

THEOREM 114′. *If a is a point of a decomposable connexe M, there exists a domain D containing a such that M is not an irreducible connexe closure from a to any point of D.*

THEOREM 115′. *If a and b are two distinct points, M is an irreducible connexe closure from a to b, and T is a proper connexe subclosure of M containing b, then $M-M\cdot\overline{T}$ is connected.*

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