The corresponding expression for what I call the type A derivative—based on another, but equally logical definition—is merely the first term of the above expression.

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ON THE ASYMPTOTIC LINES OF A RULED SURFACE

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Many mathematicians have studied the surfaces every *asymptotic curve* of which belongs to a linear complex. I will here be content with the results given on pages 112–116 and 266–288 of a treatise[1] written by myself and Professor A. Cech. This treatise gives (p. 113) a very simple proof of the following theorem:

If every non-rectilinear asymptotic curve of a ruled surface S belongs to a linear complex, all these asymptotic curves are projective to each other.

We will find all the ruled surfaces, the non-rectilinear asymptotic curves of which *are projective to each other*, and prove conversely that *every one of these asymptotic curves belongs to a linear complex*. If c, c' are two of these asymptotic curves and if A is an arbitrary point of c, we can find on c' a point A' such that the straight line AA' is a straight generatrix of S. The projectivity, which, according to our hypothesis, transforms c into c', will carry A into a point A_1 of c'. We will prove that *the two points A' and A_1 are identical*; but this theorem is not obvious and therefore our demonstration cannot be very simple. The generalization to nonruled surfaces seems to be rather complicated: and we do not occupy ourselves here with such a generalization.

If the point $x=x(u, v)$ generates a ruled surface S, for which $u=$const. and $v=$const. are asymptotic curves, we can suppose (loc. cit., p. 182)

$$x = y + uz \tag{1}$$

in which y and z are functions of v. More clearly, if x_1, x_2, x_3, x_4 are homogeneous projective coordinates of a point of S, we can find eight functions y_i and z_i of v such that

$$x_i = y_i(v) + uz_i(v), \qquad i = 1, 2, 3, 4. \tag{1_{bis}}$$

From the general theory of surfaces, it is known (loc. cit., p. 90) that

[1] *Geometria Proiettiva Differenziale*, Bologna, Zanichelli.

we can find five functions θ, β, γ, p_{11}, p_{22} of u, v such that

$$\left\{\begin{aligned} x_{uu} &= \theta_u x_u + \beta x_v + p_{11}x, \\ x_{vv} &= \gamma x_u + \theta_v x_v + p_{22}x, \end{aligned}\right. \tag{2}$$

$$x_u = \frac{\partial x}{\partial u}, \quad \theta_u = \frac{\partial \theta}{\partial u}, \quad x_{uu} = \frac{\partial^2 x}{\partial u^2}, \cdots ;$$

$$x = x_i; \quad i = 1, 2, 3, 4; \quad x = y + uz.$$

Since now $x_{uu}=0$, the former of these equations becomes

$$0 = \theta_u x_u + \beta x_v + p_{11}x,$$

and therefore (since the points x, x_u, x_v are independents):

$$\theta_u = \beta = p_{11} = 0.$$

Equation (2) becomes

$$y_{vv} + uz_{vv} = \gamma z + \theta_v(y_v + uz_v) + p_{22}(y + uz).$$

And, by differentiating two times with respect to u,

$$0 = \frac{\partial^2 p_{22}}{\partial u^2} y + \frac{\partial^2(\gamma + up_{22})}{\partial u^2} z.$$

Therefore

$$0 = \frac{\partial^2 p_{22}}{\partial u^2} = \frac{\partial^2}{\partial u^2}(\gamma + up_{22})$$

and we can write

$$p_{22} = A + Bu, \qquad \gamma + up_{22} = C + Du, \tag{3}$$

$$\gamma = (C + Du) - \mu(A + Bu), \tag{4}$$

in which A, B, C, D are functions only of v. We can multiply the x_i or, what is the same, the y_i and the z_i by a factor of proportionality (function *only of* v) such that $\theta=\text{const.}$, and $\theta_v=0$ (or that the determinant of the y_i, z_i, $y_i' = \partial y_i/\partial v$, $z_i' = \partial z_i/\partial v$ becomes a constant). The second equation of (2) becomes

$$x_{vv} = \gamma z + px, \qquad p = p_{22} = A + Bu,$$

and, by differentiating with respect to u,

$$z_{vv} = Dz + By = (D - uB)z + Bx.$$

Therefore

$$\gamma z = x_{vv} - px, \qquad p = A + Bu = p_{22},$$

$$\frac{\partial^2}{\partial v^2}\frac{x_{vv} - px}{\gamma} - (D - uB)\frac{x_{vv} - px}{\gamma} - Bx = 0;$$

or

$$x'''' - 2\frac{\gamma'}{\gamma}x''' + \left[2\left(\frac{\gamma'}{\gamma}\right)^2 - \frac{\gamma''}{\gamma} - A - D\right]x'' + 2\frac{pq' - p'q}{\gamma}x'$$

$$(5) \qquad + \left[2\frac{\gamma'}{\gamma}\frac{p'q - pq'}{\gamma} + \frac{pq'' - p''q}{\gamma} + AD - BC\right]x = 0,$$

$$p = A + Bu, \qquad q = C + Du, \qquad \gamma = q - up,$$

$$p' = \frac{\partial p}{\partial v}, \qquad \gamma' = \frac{\partial \gamma}{\partial v}, \qquad x' = \frac{\partial x}{\partial v}, \cdots .$$

This is the differential equation which defines the asymptotic curves $u = \text{const}$. If we put $x = X\gamma^{1/2}$, this equation becomes

$$X'''' + lX'' + mX' + nX = 0$$

in which

$$l = 2\frac{\gamma''}{\gamma} - \frac{5}{2}\left(\frac{\gamma'}{\gamma}\right)^2 - (A + D), \qquad n = -\frac{35}{16}\left(\frac{\gamma'}{\gamma}\right)^4 + \frac{r}{\gamma^3},$$

$$m = 2\frac{\gamma'''}{\gamma} - 7\frac{\gamma'\gamma''}{\gamma^2} + 5\left(\frac{\gamma'}{\gamma}\right)^3 + 2\frac{pq' - qp'}{\gamma} - (A + D)\frac{\gamma'}{\gamma}$$

(r is a polynomial of the variable u). The projective invariants (or covariants) of the curve defined by this equation are

$$Udv^3, \qquad V_1dv^2, \qquad Wdv^4.$$

We have put

$$U = l' - m = \frac{\epsilon}{\gamma}, \qquad [\epsilon = \{(A' - D')C - (A - D)C'\}$$

$$+ 2(CB' - BC')u + \{(A' - D')B - B'(A - D)\}u^2],$$

$$W = 20l'' - 50m' - 9l^2 + 100n = k\left(\frac{\gamma'}{\gamma}\right)^4 + \frac{R}{\gamma^3},$$

$$k = \text{const.} = 175 \neq 0; \; R \text{ a polynomial of } u,$$

and (if $U \neq 0$)

$$V_1 = 6[\log U]'' - \left(\frac{U'}{U}\right)^2 - \frac{36}{5}l, \qquad U' = \frac{\partial U}{\partial v}.$$

If $U=0$, the curve belongs to a linear complex; if $U\neq 0$,

$$\frac{W^3}{U^4} = \frac{(k\gamma'^4 + R\gamma)^3}{\gamma^8\epsilon^4}$$

is a projective invariant. If all the asymptotic curves $u=\text{const.}$ are projective to each other, this ratio must not be dependent upon u. And therefore the values of u, for which $\gamma=0$, must also satisfy the equation $\gamma'=0$. Therefore

$$\frac{C'}{D'} = \frac{D' - A'}{D - A} = \frac{B'}{B}$$

or

$$\gamma = C + (D - A)u - Bu^2 = V(c + bu + au^2)$$

(V function of v; a, b, $c=\text{const.}$) Therefore $\epsilon=0$, $U=0$ and every asymptotic curve $u=\text{const.}$ belongs to a linear complex. In this case

$$\frac{\gamma'}{\gamma} = \frac{V'}{V}, \qquad \frac{\gamma''}{\gamma} = \frac{V''}{V}, \qquad B' = \frac{V'}{V} B,$$

$$\frac{pq' - p'q}{\gamma} = \frac{p(q' - up') - p'(q - up)}{\gamma} = p\frac{\gamma'}{\gamma} - p'$$

$$= (A + Bu)\frac{V'}{V} - \left(A' + Bu\frac{V'}{V}\right) = \frac{AV' - A'V}{V}\cdot$$

And analogously

$$\frac{pq'' - p''q}{\gamma} = \frac{AV'' - A''V}{V}\cdot$$

Therefore no one of the coefficients of (5) is dependent upon u, and consequently we can suppose that the projectivity, which carries an asymptotic curve $u=\text{const.}$ into another, carries every point of the former into that point of the latter which belongs to the same rectilinear generatrix of the surface (because the corresponding value of v is not changed by this projectivity).

We have in this manner completely demonstrated the stated theorems.

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