irreducible subspaces. This paper is concerned with a study of the law of combination satisfied by the indices of the tensor associated with each diagram or partition of the indices of the arbitrary tensor. It is noted that corresponding to each partition there can be associated a specialization of C. M. Cramlet's general invariant tensor (Tôhoku Mathematical Journal, vol. 28 (1927), pp. 242–250), and that an arbitrary tensor can be decomposed by multiplying an appropriate numerical tensor identity by it, and then contracting. (Received November 25, 1940.)

11. R. W. Wagner: *The differentials of analytic matrix functions.*

The notions of Hausdorff and Fréchet concerning differentials are applied to analytic matrix functions as defined by the author in a previous paper. The differential of an analytic matrix function is shown to have a simple form when it is written in terms of the characteristic roots and partial idempotent and partial nilpotent elements of the argument. The function enters the differential through divided difference quotients. This simple form of the differential is used to deduce local properties of the mapping defined by the function. (Received November 23, 1940.)

12. Morgan Ward: *The fundamental theorem of arithmetic.*

A set of necessary and sufficient conditions for the fundamental theorem of arithmetic to hold in a semi-group is obtained from the theory of residuated lattices and a new inductive proof of the theorem is given for the lattice of positive integers. (Received October 30, 1940.)

13. Hermann Weyl: *Theory of reduction for arithmetical equivalence.* II.

Instead of the arithmetically refined method of reduction used in the first paper, the author now resorts to a rougher method which also goes back to Minkowski, and works without the assumption that the class number of ideals is 1. Its generalization to algebraic number fields F with several infinite prime spots is due to Siegel and P. Humbert (Commentarii Mathematici Helvetici, vol. 12 (1939–1940), pp. 263–306). The author follows the same geometric approach as before, including all classes of lattices over F and adding to the case of a field F that of a field quaternion algebra with totally positive norm over a totally real field. (Received November 23, 1940.)

Analysis

14. Warren Ambrose: *Representation of ergodic flows.* Preliminary report.

A flow is a one-parameter group T_t $(-\infty < t < \infty)$ of measure preserving transformations of a space Ω into itself. It is measurable if the function T_tP is a measurable function on $T \times \Omega$, where T denotes the real line taken with Lebesgue measure. For a measurable ergodic flow it is shown that a necessary and sufficient condition that a group of unitary operators U_t defined by $U_tf(P) = f(T_tP)$ have an eigenvalue (other than the trivial eigenvalue 1) is that the flow be isomorphic (with respect to measure properties) to a flow built on a measure preserving transformation (for definition of such a flow see abstract 46-11-446), thus showing that if an ergodic flow has an eigenvalue the measure on Ω must be the direct product measure of a cross section measure with Lebesgue measure along the trajectories of the flow. It is also shown that if the flow has no eigenvalues but satisfies certain conditions which are stronger than measurability then the measure on Ω is such a direct product measure. (Received November 18, 1940.)

15. R. P. Boas and D. V. Widder: *Completely convex functions.* Preliminary report.

A function $f(x)$ is defined to be completely convex in an interval (a, b) if it has even derivatives which satisfy the relation $(-1)^n f^{(2n)}(x) \geqq 0$ there. For example, the functions $\sin x$ and $\cos x$ have the property in the intervals $(0, \pi)$ and $(-\pi/2, \pi/2)$ respectively. It was shown by Widder (Proceedings of the National Academy of Sciences, November, 1940) that such a function must be entire of order at most unity (and type at most $\pi/(b-a)$ if the order is one). A simplified proof of this result is given, and an integral representation for completely convex functions is obtained. Inversion formulas for the integral involved in this representation are discussed. (Received November 23, 1940.)

16. Russell Cowan: *A method of solving a linear difference equation with polynomial coefficients of degree m.*

Given a linear difference equation Δ of order p with polynomial coefficients of degree m, a linear differential equation D is found whose recursion relation yields Δ. The order of D is the smaller of p and m. Under slight restrictions on the coefficients of Δ, the differential equation has $p+2$ regular singular points. The case $p=2$ is studied in detail. By transforming D to Heun's equation, the solution of Δ is readily obtained. In special cases one of the singular points of D is not regular. But even under these circumstances, Δ can be solved by utilizing the characteristic exponent λ which is found by transforming D into a Riccati equation. (Received November 18, 1940.)

17. J. H. Curtiss: *Degree of polynomial approximation on a lemniscate.*

It is known that a function $F(z)$ analytic and single-valued interior to the lemniscate Γ: $|\omega(z)| = \mu$, $\omega(z) = (z-\alpha_1)(z-\alpha_2) \cdots (z-\alpha_\lambda)$, can be expanded in a series of the form $\sum_0^\infty q_\nu(z)[\omega(z)]^\nu$ convergent to $F(z)$ for $|\omega(z)| < \mu$, where the functions $q_\nu(z)$ are polynomials of degree less than λ. Let $S_n(z; F)$ denote the nth partial sum of this series, let $S_n^{(r)}(z; F)$ denote the nth Cesàro mean of order r, $0 < r \leqq 1$, and let $J_n(z; F)$ denote the nth Jackson mean. The following theorem is proved: If C consists of one or several of the closed contours of Γ, and if $G(z)$ is a function analytic interior to C, continuous in the closed region or regions, and satisfying a Lipschitz condition of order η on C, $0 < \eta \leqq 1$, then there exists a function $F(z)$ analytic interior to Γ such that, uniformly on and interior to C, we have $S_n(z; F) - G(z) = O(n^{-\eta/m} \log n)$, $S_n^{(r)}(z; F) - G(z) = O(n^{-r}) + O(n^{-\eta/m})$, $r \neq \eta/m$, $= O(n^{-r} \log n)$, $r = \eta/m$, $J_n(z; F) - G(z) = O(n^{-\eta/m})$, where m is the order of the multiple point on C of highest order. If $C = \Gamma$, then $F(z) \equiv G(z)$. Examples are given to show that these estimates cannot be improved. (Received November 27, 1940.)

18. J. H. Curtiss: *Trigonometric interpolation by means of the complex Lagrange polynomial.*

Let $L_n(\theta; F)$ be the unique polynomial in $e^{i\theta}$ of degree at most $n-1$ which coincides with a given function $F(\theta)$ in the points $\theta = 2\pi k/n$, $k = 1, 2, \cdots, n$. Trivial examples show that even if $F(\theta)$ is analytic for all θ, the sequence $\{L_n(\theta; F)\}$ may diverge for all $\theta \not\equiv 0$, mod 2π. This paper studies the convergence of $\{L_n(\theta; F)\}$ for functions $F(\theta)$ which are continuous and are the boundary value functions of functions $f(z)$, $z = re^{i\theta}$, analytic and of class H for $|z| < 1$. The following results are typical: (a) If the Fourier series for $F(\theta)$ converges absolutely for all θ, then $L_n(\theta) \to F(\theta)$ uniformly; (b) if $F(\theta)$ is of bounded variation in an open neighborhood of θ_1, then $L_n(\theta) \to F(\theta)$ uniformly in a closed subneighborhood of θ_1; (c) $L_n(\theta; G_n) \to F(\theta)$ uniformly, where $G_n(\theta) = (n/2\pi) \int_\theta^{\theta+2\pi/n} F(t)dt$; (d) $L_n(\theta; G_n) - F(\theta) = O(n^{-\alpha})$ if $F(\theta)$ satisfies a Lipschitz condition with exponent $\alpha < 1$. These results are then generalized to the case of interpolation on a Jordan curve C in the manner of the author's paper in the Transactions of this Society, vol. 38 (1935), p. 458. (Received November 27, 1940.)

19. L. L. Dines: *On the mapping of quadratic forms.*

If $P(z)$ and $Q(z)$ are real quadratic forms in n variables $z_1, z_2, \cdots, z_n$, the transformation $x = P(z)$, $y = Q(z)$ maps the n-dimensional z-space into a set of points $\mathfrak{M}$ of the xy-plane. This paper obtains properties of the map $\mathfrak{M}$, and from them deduces certain results relative to the pair of forms $P(z)$ and $Q(z)$. Among the results is the theorem proved by Albert (this Bulletin, vol. 44 (1938), p. 250) and by Reid (this Bulletin, vol. 44 (1938), p. 437). The paper will appear in the Bulletin. (Received November 14, 1940.)

20. Jesse Douglas: *Solution of the inverse problem of the calculus of variations.*

A solution is presented of the classical problem: given a family of curves in 3-space, as defined by differential equations $y'' = F(x, y, z, y', z')$, $z'' = G(x, y, z, y', z')$; determine the existence or nonexistence of a calculus of variations problem of the form $\int \phi(x, y, z, y', z')dx = \text{minimum}$ having the prescribed curve family for the totality of its extremals, and, in the affirmative case, to find the most general form of the function ϕ. Preliminary notes stating the results and describing the methods used have already appeared in the Proceedings of the National Academy of Sciences, vol. 25 (1939), pp. 631–637, and vol. 26 (1940), pp. 215–221. A detailed paper will be published in the Transactions of this Society. (Received November 20, 1940.)

21. J. J. Eachus: *Classification of solutions and of pairs of solutions of $y''' + 2p(x)y' + p'(x)y = 0$ by means of boundary conditions.*

G. D. Birkhoff (Annals of Mathematics, (2), vol. 12 (1911), p. 103) has shown that the differential equation $y''' + 2p(x)y' + p'(x)y = 0$ has three types of solutions according to the number and nature of zeros of the solution. He has further shown that there are only a limited number of possible configurations of the zeros of two solutions of the equation. These same conclusions are deduced in this paper in a new manner, one which leads to a method for determining the "type" of a solution y_1 from the values of y_1 and its first two derivatives at any point x_0, and a method for determining the configuration of the zeros of y_1 and y_2 from the values of y_1 and y_2 and their first two derivatives at any point x_0. (Received November 25, 1940.)

22. Samuel Eilenberg: *Linear measure and convexity.*

Let X be a continuum with the metric ρ. If the linear measure $L(X, \rho)$ is finite then there is a metric ρ' equivalent with ρ and such that (1) ρ' is convex (2) $\rho' \geqq \rho$ (3) $L(X, \rho') = L(X, \rho)$. The metric ρ' is unique. (Received October 24, 1940.)

23. K. W. Folley: *A property of a simply ordered set.*

It is shown in this paper that a method of proof which was employed by Sierpiński to prove a theorem concerning the decomposition of real numbers may be used to prove an analogous theorem concerning the decomposition of a simply ordered set which contains an everywhere dense η_α subset. This result leads to a proposition which is proved to be equivalent to the generalized hypothesis of the continuum. (Received October 25, 1940.)

24. H. L. Garabedian: *Relations between hypergeometric methods of summation.*

The definition of hypergeometric summability $(H, \alpha, \beta, \gamma)$ is extended and new inclusion and equivalence relations between hypergeometric methods of summation are obtained. The relation $(H, \alpha_1, 1, \alpha_1+\gamma)\approx(H, \alpha_2, 1, \alpha_2+\gamma)$, $\Re(\alpha_1, \alpha_2, \gamma)>0$, is probably the most significant result established. Corollaries to this result are $(H, \alpha, 1, \alpha+\beta)\approx(C, \beta)$, $\Re(\alpha, \beta)>0$; $(H, \alpha, 1, \gamma)\approx(C, \gamma-\alpha)$, $\Re(\alpha, \gamma, \gamma-\alpha)>0$. These statements imply that, given any hypergeometric method of summation of the type $(H, \alpha, 1, \gamma)$, there exists a Cesàro method of summation equivalent to it; and, conversely, given any Cesàro method of summation (C, β) there exists a single infinitude of hypergeometric methods of summation of the type $(H, \alpha, 1, \alpha+\beta)$ equivalent to it. (Received November 20, 1940.)

25. J. W. Green: *A special type of conformal map.*

Let G_z be the unit circle in the z plane, O an open set of points on the circumference of G_z, and F its complement with respect to the circumference. The set F is supposed to be of positive measure. In the case $\bar{O}-O$ is of zero measure, there exists a function $w=w(z)$ which maps G_z conformally on the interior of the unit circle G_w with certain radial slits deleted. This transformation carries an interval of O into all or part of a radial slit, and carries points of F into points of $|w|=1$, almost everywhere on F. (Received November 25, 1940.)

26. R. E. Greenwood: *Hankel and other extensions of Dirichlet series.*

This paper considers series of the form $g(s)=\sum_{n=1}^{\infty}a_n \exp(-\lambda_n s)G(\lambda_n s)$. If $G(z)\equiv 1$, then the series reduces to the well known Dirichlet series. If $G(z)$ is analytic in the right half-plane, is bounded in the half-plane to the right of $z_0>0$ and has a certain asymptotic expansion, the series above will be convergent in a half-plane, and also absolutely convergent in a half-plane. The abscissae of convergence are the same as the abscissae for the corresponding Dirichlet series. Summation formulae involving the first m coefficients are developed, and these reduce to Perron's formula for Dirichlet's series when $G(z)\equiv 1$. As a particular case such a function $G(z)$ is chosen so that $g(s)$ is a series of Hankel functions of the first kind of fixed order ν on the imaginary axis, and a few special results are obtained for this series. (Received November 14, 1940.)

27. H. J. Hamilton: *On monotonic and convex solutions of certain difference equations.*

The principal result of this paper is reduction of the problems of existence and of uniqueness (excepting an additive constant) of continuous, convex solutions U of the equation $U(x+1)-U(x)=G(x)$ to the same problems for monotone non-decreasing solutions of a related equation of the same form. This reduction extends itself partially to linear difference equations of higher order with constant coefficients. Our conclusions are applied to some results of Fritz John. (Received September 28, 1940.)

28. L. B. Hedge: *Transformations of multiple Fourier series.* Preliminary report.

If $(n)=(n_1, n_2, \cdots, n_k)$ is a lattice point of euclidean k-space and $(x)=(x_1, x_2, \cdots, x_k)$ is a point of euclidean k-space, and if $a_{(n)}$ and $\lambda_{(n)}$ are functions on the lattice points of k-space, we consider the series $S=\sum a_{(n)}e^{i(n\cdot x)}$ and its transform $\lambda S=\sum \lambda_{(n)}a_{(n)}e^{i(n\cdot x)}$ where $(n\cdot x)=n_1x_1+n_2x_2+\cdots+n_kx_k$. A classification, analogous to that used for single series, is defined for the series S and λS, and transformations λ which take one class into another are characterized for interesting cases. The results include and extend the known transformation theory for single trigonometric series. The spherical summation of Bochner and a recent moment problem solution by the author are the principal tools used for the study. (Received December 2, 1940.)

29. A. E. Heins: *On the solution of partial difference equations. Two interval "boundary conditions."* Preliminary report.

The solution of the partial difference equation $f(x+1, t)+f(x-1, t)=2f(x, t+1)$ is considered under the following initial and boundary conditions: For $[t]=0, f(x, t)$ is prescribed, $af(x-[x], t)+f(x-[x]+1, t)=0$ and $bf(x-[x]+p, t)+f(x-[x]+p+1, t)=0$. The partial difference equation is first reduced to an ordinary difference equation with the Laplace transform and the resulting ordinary difference equation solved under the given boundary conditions. The solution depends on a finite expansion of sines and cosines of multiples of angles which are roots of a certain transcendental equation. The nature of this rather interesting expansion will be discussed at a later date. (Received November 18, 1940.)

30. M. R. Hestenes: *Extension of the range of a differentiable function.*

In the present paper two methods of extending the range of a differentiable function are given. The first method is essentially a reflection across boundaries. By this method it is shown that a function $f(x_1, \cdots, x_n)$ of class C^m (m finite) on a closed set A with a suitable boundary can be extended to be of class C^m over the whole of euclidean n-space. The second method is similar to one used by Whitney (Transactions of this Society, vol. 36 (1934), pp. 63–89) to show that a function of class C^m on an arbitrary closed set A can be extended to be of class C^m over the whole space and to be of class C^∞ at the points not in A. (Received November 26, 1940.)

31. Einar Hille: *A class of differential operators of infinite order.* II.

Let $D_zw=P_0(z)w''+P_1(z)w'+P_2(z)w$. Let $G(z)$ be an entire function of order $\frac{1}{2}$ and minimal type. Then the operator $G(D_z)$ preserves holomorphism in any domain where the P's are holomorphic. The condition on $G(z)$ is also necessary for most operators D_z of practical importance. The equation $G(D_z)\cdot U=0$ can be studied by the methods of Ritt and Valiron. If the P's are single-valued and $a_1, \cdots, a_m$ are the singularities of $D_zw=0$, let R be an unbounded non-ramified covering surface of the plane punctured at $a_1, \cdots, a_m$ on which all solutions of $(D_z-\lambda_n)w=0$ are single-valued, where $\{\lambda_n\}$ are the roots of $G(z)=0$. $U(z)$ is single-valued on R. Its domain of existence E is such that any maximal component of $R-E$ has at least one point a_k

as limit-point. If R is simply-connected, so is E. $U(z)$ has a unique Ritt series in terms of solutions of $(D_z-\lambda_n)w=0$. If $P_0=P_1=1$, P_2 entire, and the λ_n suitably restricted, this series converges in E, and E is convex. The results extend to differential operators D_z of order n. (Received October 5, 1940.)

32. Fritz John: *On the character of solutions of hyperbolic equations.* Preliminary report.

Let $u(x_1, \cdots, x_n)$ be a solution of a linear, normal-hyperbolic differential equation of second order with analytic coefficients. The solution u is shown to be "pseudo-analytic" in the following sense: Given a family of "time-like" analytic curves C_α depending on a parameter α, all C_α having the same two endpoints, then $\int udt$ is an analytic function of α, t being an analytic parameter on C_α. It follows that the values of u on any time-like two-dimensional analytic manifold, interior to the domain of regularity of u, are not independent. This is a generalization of results previously obtained by the author for a special equation (Mathematische Annalen, vol. 111, pp. 541–555). The proof makes use of the form of the solution obtained by J. Hadamard in his *Lectures on Cauchy's Problem.* (Received November 20, 1940.)

33. J. P. LaSalle: *A note on pseudo-normed linear spaces.*

The equivalence of speudo-normed linear spaces (p.l.s.'s) and linear topological spaces (l.t.s.'s) has been shown by Hyers. (See Duke Mathematical Journal, vol. 5 (1939), pp. 628–634.) In the present paper l.t.s.'s are studied as characterized in terms of the pseudo-norm. It is shown that the existence of an open convex set containing the zero element and properly contained in a p.l.s. T is a necessary and sufficient condition for the existence of a non-null linear functional on T. An example is given of a p.l.s. on which no non-null linear functional can be defined. Also the set of all linear functions on a p.l.s. T to a p.l.s. T' is shown to be a pseudo-normed linear space. If T' is convex, then the p.l.s. of all linear functions on T to T' is itself convex. Hence the set of all linear functionals on a p.l.s. T is a convex p.l.s. If a p.l.s. T is locally bounded, then the set of all linear functionals on T is shown to be a normed linear space. (Received October 22, 1940.)

34. J. P. LaSalle: *Pseudo-normed linear sets over valued rings.* I.

In this paper a generalization of linear spaces is given by replacing real number multipliers by multipliers taken from a valued ring. A valued ring is defined to be a ring A with a unity element such that there is defined on A a real-valued function $M(\alpha)$ where (1) $M(\alpha)\geqq 0$ for all $\alpha\in A$; (2) $M(\alpha\beta)\leqq M(\alpha)M(\beta)$; (3) $M(\alpha+\beta)\leqq M(\alpha)+M(\beta)$; (4) $M(-1)=1$; (5) $M(\alpha)>1$ for some $\alpha\in A$. The set of all formal polynomials with coefficients taken from a ring with a unity element can be shown to be a valued ring, which indicates somewhat the generality of valued rings. A linear set T over a valued ring A is said to be pseudo-normed w.r.t. a strongly partially ordered set D if there exists a real-valued function $n(x, d)$ on TD such that (1) $n(x, d)\geqq 0$; $n(x, d)\leqq 1$ for all $x\in T$ implies $x=\Theta$; (2) $n(\alpha x, d)\leqq M(\alpha)n(x, d)$; (3) given $d\in D$ there exist $e\in D$ such that $n(x, e)\leqq 1$, $n(y, e)\leqq 1$ implies that $n(x+y, d)\leqq 1$; (4) given $x\in T$ and $d\in D$ there exist $\alpha\in A$ and $y\in T$ such that $x=\alpha y$, $n(y, d)\leqq 1$; (5) given $\alpha\in A$ and $d\in D$ there exist $e\in D$ such that $n(x, e)\leqq 1$ implies that $n(\beta x, d)\leqq 1$ for all $M(\beta)\leqq M(\alpha)$; (6) $e\geqq d$ implies that $n(x, e)\geqq n(x, d)$. Another characterization of T is given in terms of a neighborhood system. The topological properties of such a space T are then studied. Later a generalized concept of linear functions and differentials will be introduced for such spaces. (Received October 22, 1940.)

35. Walter Leighton: *On the convergence of continued fractions.*

Let $z_2, z_3, \cdots$ be any bounded sequence of complex numbers such that $R(z_n) \geqq \epsilon > 0$ $(n=2, 3, \cdots)$, where ϵ may be taken arbitrarily small. The continued fraction $1+K_1^\infty [a_n/1]$ converges, where $a_n=(z_{n-1}+1/2)(z_n+1/2)$ $(n=2, 3, \cdots;$ $z_1=1)$. (Received November 18, 1940.)

36. J. Marcinkiewicz and Antoni Zygmund: *On the behavior of trigonometric and power series.*

A point set Z on the plane is said to be of circular structure with center z_0 if whenever a point ζ belongs to Z, so does the whole circle $|z-z_0|=|\zeta-z_0|$. Given a power series $\sum_0^\infty c_n z^n$, let $\sigma_n^\alpha(\theta)$ denote the αth Cesàro means of the series $\sum_0^\infty c_n e^{in\theta}$ and let $L^\alpha(\theta)$ denote the set of the limit points of the sequence $\sigma_0^\alpha(\theta), \sigma_1^\alpha(\theta), \cdots, \sigma_n^\alpha(\theta), \cdots$. The chief result of the paper may be stated as follows: If for every θ belonging to a set E of positive measure the series $\sum_0^\infty c_n e^{in\theta}$ is summable $(C, \alpha+1)$ $(\alpha > -1)$ to sum $t(\theta)$, then for almost every θ of E the set $L^\alpha(\theta)$ is of circular structure with center $t(\theta)$. (Received October 31, 1940.)

37. R. S. Martin: *Minimal positive harmonic functions.*

The idea of a minimal positive harmonic function arises as follows. The Poisson-Stieltjes integral formula for a sphere depends upon (1) a family of positive harmonic functions (namely the kernel $F(S, P)$ where the point S lies on the boundary and is considered as a parameter), and (2) a family of linear operations (namely integration with respect to mass-distributions on the boundary). Given then a general bounded domain D, one may ask for suitable generalizations of the situations just described. The idea of the present paper is to consider, as a generalization of the family $F(S, P)$ corresponding to the sphere, the family of those functions which are *minimal*, positive and harmonic in D. A function $u(P)$, positive and harmonic in D, is called *minimal* for D if it dominates there no positive harmonic functions, except those of the form $cu(P)$, where $c>0$ is a constant. The purpose of the paper is to discuss the theory of representations of harmonic functions based on this idea. (Received October 22, 1940, from Tibor Radó.)

38. C. N. Moore: *On the Cesàro and Abel-Poisson summability of the differentiated double Fourier series.*

It has been shown that at points where $f(x)$ has a derivative or generalized derivative the differentiated Fourier series will be summable (C, r) for $r>1$ or will be summable by the Abel-Poisson method. In the present paper these results are extended to the partial derivatives or generalized partial derivatives of functions of two variables and the corresponding differentiated double Fourier series. The result concerning Abel-Poisson summability appears as a corollary of the result concerning Cesàro summability by use of general convergence factor theorems. (Received November 27, 1940.)

39. G. D. Nichols: *A sufficient condition for Cesàro summability.*

The series $\sum f(r) \cos rx$ and $\sum f(r) \sin rx$ are shown to be summable (C, k) provided the $\Delta^k f(r)$ is a monotone null sequence. In case $f(r)$ is a polynomial of degree $k-1$, a closed expression is obtained for the Cesàro sum. (Received November 23, 1940.)

40. I. E. Perlin: *Indefinitely differentiable functions of several real variables.*

In the present paper the author considers indefinitely differentiable functions of several real variables. Let $M_{ij\cdots k}=\text{l.u.b.}\ |\partial^{i+j+\cdots+k}f(x_1, x_2, \cdots, x_n)/\partial x_1^i \partial x_2^j \cdots \partial x_n^k|$ on $0 \leqq x_p \leqq a_p$. Sufficient conditions that there exist a function $f(x_1, x_2, \cdots, x_n)$ with prescribed $M_{ij\cdots k}$ are established. (Received November 25, 1940.)

41. Arthur Sard: *The measure of the critical values of differentiable maps of euclidean spaces.*

Consider the map $y^j=f^j(x^1, \cdots, x^m)$, $j=1, 2, \cdots, n$, of a region R of euclidean m-space into part of euclidean n-space. Suppose each of the functions f^j is of class at least C^q in R ($q \geqq 1$). A critical point of the map is a point in R at which the matrix of first derivatives, (f_i^j), $i=1, 2, \cdots, m$; $j=1, 2, \cdots, n$, is of less than maximum rank. A critical value y is the image $f(x)$ of a critical point x. The following theorem is proved: If $m \leqq n$, the set of all critical values is of m-dimensional measure zero (in the sense of Hausdorff-Saks) without further hypothesis on q; if $m>n$, the set of all critical values is of n-dimensional measure zero providing $q \geqq m-n+1$. For $n=1$, this result specializes to a known theorem, surmised by Marston Morse and proved for all m by A. P. Morse (Annals of Mathematics, (2), vol. 40 (1939), pp. 62–70). The proof of the theorem of the present paper depends in part on a result of A. P. Morse. The hypothesis on q cannot be weakened, as can be seen from an example due to H. Whitney (Duke Mathematical Journal, vol. 1 (1935), pp. 514–517). (Received November 25, 1940.)

42. H. M. Schwartz: *Convex functions and the law of the mean.*

Let S denote the class of functions of a real variable for which the law of the mean holds throughout a given interval, and let T denote the class of functions which are either strongly convex or strictly concave in that interval. The classical conditions insuring that the function f be in S are for f of T necessary as well as sufficient. The class U common to both T and S is uniquely determined by the requirement that the θ-function occurring in the law of the mean be a single-valued function of its arguments. This θ-function for f of U is studied both as to its dependence on f and as to its dependence on the variable and its increment. It is shown, for instance, that if $f_n \in S$ ($n=1, 2, \cdots$), sequence f_n' converges continuously to f' in the interval, and $f \in U$, then $\theta(f_n)$ converges to $\theta(f)$ in the domain of its arguments. Most of the results are extended to functions of many variables. (Received November 20, 1940.)

43. H. M. Schwartz: *Sequences of Stieltjes integrals.*

The paper contains a study of some of the properties of function sequences of the form $\int_a^x f dg_n = s_n$ ($n=1, 2, \cdots$), $a<x \leqq b$ (a, b finite or infinite), where f, g_n are bounded functions for which the integrals exist in the sense of Riemann-Stieltjes. When g_n are of bounded variation an obvious sufficiency condition for the convergence of s_n in (a, b) can be stated in terms of the integrability conditions of f with respect to g_n, but it is of interest to find such conditions in terms of the convergence properties of the sequence g_n. This problem is solved here only partially. Assuming that g_n are uniformly of bounded variation in (a, b), the convergence of s_n is proved for special classes of f and ordinary convergence of the sequence g_n, and generally for special modes of convergence of this sequence. A typical result is as follows: s_n converges in

(a, b) (assumed finite) if $g_n \to g_0$ and if $\int f dg_n$ $(n=0, 1, 2, \cdots)$, $\int f dv$ exist, where v is any of the limit functions of the sequence v_n, $v_n(x)$ being the total variation of g_n in (a, x). (Received November 25, 1940.)

44. Hyman Serbin: *Upper bounds for the remainder of certain power series.*

This note exemplifies a method of obtaining an upper bound of the remainders of power series in which the coefficients are connected by a recurrence relation with a constant number of terms. The function considered is wave mechanical $\exp(-ix) \cdot F(m+1-ai; 2m+2; 2ix)$ where $F(\alpha; \gamma; z)$ is the confluent hypergeometric function. A majorant of the remainder, suggested by the recurrence relation, is expressed in closed form. The upper bound so obtained is not explicitly given in terms of the parameters but depends on two successive coefficients. However, for the purpose of computation, the result is satisfactory. The above results were obtained in the course of work done by the Project for the Computation of Mathematical Tables, conducted by the Work Projects Administration of New York City. (Received October 30, 1940.)

45. W. S. Snyder: *On independence of the path for line integrals of continuous functions.*

Let $p(x, y)$ and $q(x, y)$ be defined and continuous in an oriented rectangle R of the xy-plane. A necessary and sufficient condition that $\int pdx+qdy$ be independent of the path in R is derived in terms of the double integrals of the functions p and q. The author also formulates and answers a more general form of a question raised by Menger (Proceedings of the National Academy of Sciences, vol. 25 (1939), p. 623), showing that the conditions of Menger and Fubini (Proceedings of the National Academy of Sciences, vol. 26 (1940), p. 199) cannot be satisfied for "undotted" nets. (Received November 23, 1940.)

46. Abraham Spitzbart: *Approximation in the sense of least pth powers by polynomials with a single auxiliary condition of interpolation.*

Let $F_0(z)$ be the function, analytic interior to the analytic Jordan curve C, of class E_p $(p>0)$ in that region, and satisfying the condition $F_0(\alpha)=A$ ($z=\alpha$ a point interior to C, A an arbitrary contant), which minimizes the integral $\int_C |F(z)|^p |dz|$. Let $P_n(z)$ be the minimizing polynomial of degree n for the integral $\int_C |P_n(z)|^p |dz|$, $P_n(\alpha)=A$. It is proved that the sequence $P_n(z)$, $n=0, 1, 2, \cdots$, converges maximally to the function $F_0(z)$ on the closed set Γ consisting of C plus its interior (for definition see J. L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*, American Mathematical Society Colloquium Publications). The results extend to the case where a suitable norm function $N(z)$ is introduced into the integral to be minimized. (Received November 25, 1940.)

47. Otto Szász: *On the partial sums of harmonic developments and related power series.*

Let T denote the class of all harmonic sine developments $H(r, \theta)=\sum C_v r^v \sin v\theta$, convergent for $0<r<1$, and non-negative for $0<\theta<\pi$. There is a largest R^n such that for every H in T, $\sum_1^n C_v r^v \sin v\theta$ is non-negative for $0<r<R_n$, $0<\theta<\pi$. Then R_n is characterized algebraically, which yields an asymptotic estimate. The result is applied to Fourier series of convex functions and to related power series. (Received November 26, 1940.)

48. A. E. Taylor: *Conjugations of complex Banach spaces.*

Let E be a complex Banach space. An operation T on E to E is called a conjugation if it is additive, and if $Taz=\bar{a}Tz$, $T^2z=z$. A set M in E is called an essentially real linear manifold (ess. r.l.m.) if it is closed under addition, and under multiplication by real numbers, and if x, ix in M implies $x=0$. An ess. r.l.m. is maximal if it is not a proper subset of any ess. r.l.m. If M is a maximal ess. r.l.m. every element of E is uniquely representable in the form $z=x+iy$, where x, y are in M. The operation $T(x+iy)=x-iy$ is a conjugation. T is continuous if and only if M is closed. Also, if T is any conjugation, the set of elements x such that $Tx=x$ is a maximal ess. r.l.m. It may be shown that every E has a maximal ess. r.l.m. The author does not know if there always exists a closed, maximal ess. r.l.m., but when one does exist it is possible to re-norm the space, without altering the topology, in such a way that $\|x+iy\| =\|x-iy\|$. (Received October 26, 1940.)

49. J. M. Thomas: *Orderly differential systems.*

Riquier's existence theorem for orthonomic systems is extended to a larger class of systems called orderly. Substantially the defining property of an orderly system is that it can be decomposed into orthonomic components. The proof of the existence theorem given is new, even for the case of orthonomic systems. Its chief feature is the elimination of the integers called cotes by Riquier. They are replaced by two systems of inequalities which it is thought arise more naturally and have simpler nature than the systems of inequalities among the cotes. (Received November 20, 1940.)

50. W. J. Trjitzinsky: *Analytic theory of singular elliptic partial differential equations.*

This work presents an extensive analytic theory of the equation $F(u)\equiv\sum a_{i,k} \cdot\partial^2u/dx_i\partial x_k+\sum b_i\partial u/\partial x_i+(c+\lambda)u=f$ $(i,\ k=1,\ 2,\ \cdots,\ m)$, where λ is a parameter and the coefficients are functions of $(x_1, \cdots, x_m)$ continuous in an open domain D, the first partials of the b_i and a number of partials of the a_i, k are continuous in D, and the functions, involved, may become infinite near the frontier of D. The investigation was suggested by certain analytic aspects of Schrödinger wave equations. The equation is transformed into an integral equation with the aid of a suitable function of the geodesic distance (formed for a suitable line element); the integral equation plays an essential role in the work, in particular, Fredholm's theory is locally applicable in D. Associated with F there is constructed a sequence of "regular" approximating homogeneous boundary value problems, (P_1), (P_2), $\cdots$. When F is self adjoint, theorems on existence of solutions, uniqueness properties and so forth are given for λ nonreal; also, for λ real, depending on sufficient "rarefication" of the totality of characteristic values formed for (P_1), (P_2), $\cdots$. An essential role is also played by an appropriate spectral theory based on (P_1), (P_2), $\cdots$. (Received November 28, 1940.)

51. H. S. Wall: *A theorem on real functions bounded in the unit circle.*

Let E denote the class of functions $e(x)$ which are analytic for $|x|<1$, real when x is real, and for which $M(e)=\text{l.u.b.}_{|x|<1}|e(x)|\leqq 1$. Let F denote the subclass of all functions $f(x)$ of E which are of the form $f(x)=\int_0^1 d\phi(u)/(1+xu)$ where $\phi(u)$ is real, bounded and monotone nondecreasing on the interval $0\leqq u\leqq 1$. Then there is a one-to-one correspondence between the functions of E and of F such that if $e(x)\leftrightarrow f(x)$, then $e(x)=[1-x-2f(z)]/[1-x+2xf(z)]$, $z=4x/(1-x)^2$, $|x|<1$. (Received November 7, 1940.)

52. J. L. Walsh and E. N. Nilson: *Approximation to an analytic function by functions analytic and bounded in a region.*

Let R be a finite sum of disjoint regions containing a closed set S in its interior, with the respective boundaries C_1 and C_0 composed of a finite number of analytic Jordan curves, each component region of R containing at least one point of S, no point $R-S$ separated by S from C_1. Define $u(z)$ as unity on C_1, zero on C_0, continuous in the extended plane, harmonic except on C_1 and C_0. If $v(z)$ is conjugate to $u(z)$ in $R-S$, then $u(z)=u(\infty)-(1/2\pi)\int_{C_1+C_0}\log|z-\zeta|\,dv(\zeta)$, for z not on C_1+C_0. This equation is used to define poles and points of interpolation for sequences of rational functions which are used (Walsh, Proceedings of the National Academy of Sciences, vol. 24 (1938)) to prove: Let $f(z)$ be analytic on S but not throughout R (a single region); for $M>0$, let $f_M(z)$ be that function analytic and of modulus not greater than M in R for which $[\max\ |f(z)-f_M(z)|,\ z \text{ on } S]$ is least. Then, for $0\leqq\sigma<\rho$, $\lim\sup_{M\to\infty}$ $[\max\ |f(z)-f_M(z)|,\ z \text{ in } \overline{R}_\sigma]^{1/\log M}=e^{(\sigma-\rho)/(1-\rho)}$, where R_σ is the set on which $0\leqq u(z)<\sigma$ and ρ is the largest number, $0<\rho<1$, such that $f(z)$ is analytic throughout R_ρ. (Received December 2, 1940.)

53. W. F. Whitmore: *Convergence theorems for functions of two complex variables. II.*

In continuation of the work of the first part of this paper (American Journal of Mathematics, vol. 62 (1940), pp. 687–696), results are obtained for convergence of a function of two complex variables in sector-domains $I^4=\sum_{|z_2|\leqq 1}I^2[\theta_2(z_2),\ \theta_1(z_2)]$ where $I^2(\theta_2,\ \theta_1)=E[\theta_2\geqq\arg z_1\geqq\theta_1]$ (superscript indices give the dimensionality of the domain in question). The limit function approached at the vertex surface $H^2=E[z_1=0,$ $|z_2|\leqq 1]$ is also examined, and shown to be an analytic function of z_2. The principal tool is the theory of harmonic measure. These sextor-domains are useful as domains of comparison for arbitrary domains whose boundary hypersurfaces contain a segment of an analytic hypersurface. (Received October 28, 1940.)

54. D. V. Widder: *Necessary and sufficient conditions for the representation of a function in Lidstone series.*

A set of polynomials $\Lambda_n(x)$ is defined as follows: The first, $\Lambda_0(x)$, is x. For each positive integer n the second derivative of $\Delta_n(x)$ is to be $\Lambda_{n-1}(x)$, and each polynomial except the first is to vanish at zero and at one. For example, $\Lambda_1(x)$ is $(x^3-x)/6$. In terms of these polynomials a Lidstone series for the functions $f(x)$ has the form $f(x)=\sum_{k=0}^{\infty}f^{(2k)}(1)\Lambda_k(x)+f^{(2k)}(0)\Lambda_k(1-x)$. Any partial sum of the series is a polynomial whose coefficients are determined so that it may coincide together with as many of its even derivatives as possible with $f(x)$ and its corresponding even derivatives at zero and one. These series, discovered by G. J. Lidstone, have been studied by H. Poritsky, J. M. Whittaker and I. J. Schoenberg. (References may be found in Schoenberg's paper, this Bulletin, vol. 42 (1936), pp. 284–288.) In the present paper the series are considered from the real point of view. A new proof of a theorem of the author is given (Proceedings of the National Academy, November, 1940) that a function $f(x)$ for which $(-1)^k f^{(2k)}(x) \geqq 0$ on an arbitrary interval is necessarily entire. By use of this result a necessary and sufficient condition that a function can be expanded in an absolutely convergent Lidstone series is obtained. (Received November 25, 1940.)

55. J. R. Woolson: *A Theory of projections in complex Banach spaces*. Preliminary report.

A projection P is defined as a linear operator on a complex Banach space B to B, such that $P^2=P$. Using an inter-space product $[F, f]$, $f \in B$, $F \in (B)$ the set of complex valued linear functionals defined on B, it is possible to prove the usual theorems concerning projections by certain analogies with the Hilbert space inner-product. (Cf. M. H. Stone, *Linear Transformations in Hilbert Space and Their Applications to Analysis*, American Mathematical Society Colloquium Publications, vol. 15.) The theory is used to show that $(1/n)\sum_{i=1}^{\infty} A^i$ converges weakly to a projection and to characterize a linear operator in terms of the linear manifolds which it leaves invariant. (Received October 28, 1940.)

Applied Mathematics

56. Harry Bateman: *Aerodynamical effects of changes in the fundamental equations*.

Simple examples indicate that a change from the elliptic to the parabolic type may not be as drastic in its effects as a change from the elliptic to hyperbolic type. The effects of the various changes that have been made in the equations of viscous flow by Prandtl, Oseen, Kármán and others are reviewed by the author. Some remarks are made also on mixtures of fluids and changes of state. Some new equations are considered and reference is made to the work of Duhem, Schutz, Silberstein, Natanson, Bjerknes and Kozlowski. Some remarks are made on the equations of Burgers, Mattioli and others which differ from the classical equations and yet exhibit some of the phenomena of turbulence. (Received November 18, 1940.)

57. M. A. Biot: *Finite difference equations applied to aircraft engine vibrations*.

The vibration amplitudes of the crankshaft are shown to satisfy a finite difference equation of the second order with constant coefficients. The frequency equation derived from the end conditions is solved numerically by an artifice which shortens considerably the time required for the evaluation of the critical speeds. (Received November 18, 1940.)

58. J. Bjerknes: *Some uses of mathematics in meteorology*.

The atmosphere exhibits tide-like wave phenomena, but the lunar component of the atmospheric tide is only about one tenth of the solar component. If the atmospheric tide is gravitational, just as the ocean tides, the atmosphere must be able to act as a vibrating system with a proper period very close to 12 solar hours and thereby give the solar component of the tide a greater amplitude than the lunar component. That vibration problem is mathematically rather well defined but still unsolved in its general form. The atmospheric disturbances responsible for the day to day variations of weather are mainly aperiodic and disorderly but at times they are quasi-periodic and to some extent amenable to mathematical treatment. The near-