ON THE MEAN VALUES OF AN ANALYTIC FUNCTION[1]

RAPHAEL M. ROBINSON

This note contains improvements on the results in two recent papers by Nehari.[2]

The first paper shows that if $f(z)$ is regular for $|z|<1$, and if the mean of $|f(z)|$ on the circle $|z|=r$ is less than or equal to 1 for each $r<1$, then the mean of $|f(z)|^2$ on $|z|=r$ is less than or equal to 1 for $r\leqq 6^{-1/2}$. We shall show that the conclusion is true for $r\leqq 2^{-1/2}$, but not always for a larger value of r. *More generally, we shall show that the mean of* $|f(z)|^p$ *on* $|z|=r$ *is less than or equal to* 1 *for* $r\leqq p^{-1/2}$ *(where* $p>1$ *is an integer), and that this result is the best possible.*

It will be sufficient to prove that if $g(z)$ is a function which is regular for $|z|\leqq 1$ and different from 0 for $|z|<1$, and such that the mean of $|g(z)|$ on $|z|=1$ is less than or equal to 1, then the mean of $|g(z)|^p$ on $|z|=r$ is less than or equal to 1 for $r\leqq p^{-1/2}$. For suppose $0<R<1$, and put

$$g(z) = f(Rz) : \prod_{\nu=1}^{n} \frac{z-\alpha_\nu}{1-\bar{\alpha}_\nu z},$$

where $\alpha_1, \alpha_2, \cdots, \alpha_n$ are the zeros of $f(Rz)$ in $|z|<1$. We note that $|g(z)|=|f(Rz)|$ for $|z|=1$, while $|g(z)|>|f(Rz)|$ for $|z|<1$. The function $g(z)$ evidently satisfies the above conditions. From the conclusion that the mean of $|g(z)|^p$ on $|z|=r$ is less than or equal to 1 for $r\leqq p^{-1/2}$, we see that the mean of $|f(Rz)|^p$ on $|z|=r$ is not greater than 1 for $r\leqq p^{-1/2}$, or that the mean of $|f(z)|^p$ on $|z|=r$ is not greater than 1 for $r\leqq Rp^{-1/2}$. The desired result follows by letting $R\to 1$.

We have to show that from the hypothesis $(1/2\pi)\int_0^{2\pi}|g(e^{i\theta})|\,d\theta\leqq 1$ the conclusion

$$\frac{1}{2\pi}\int_0^{2\pi} |g(re^{i\theta})|^p d\theta \leqq 1, \qquad \text{for } r \leqq p^{-1/2},$$

follows. Now since $g(z)\neq 0$ for $|z|<1$, we may put $g(z)=h(z)^2$, where $h(z)$ is regular for $|z|<1$. If we put

$$h(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad h(z)^p = \sum_{n=0}^{\infty} c_n z^n,$$

[1] Presented to the Society December 2, 1939.

[2] Comptes Rendus de l'Académie des Sciences, Paris, vol. 206 (1938), pp. 1943–1945; vol. 208 (1939), pp. 1785–1787. My results were obtained during a summer (1939) spent at Stanford University. The two papers mentioned were called to my attention by Professor Szegö.

and use the well known formula

$$\frac{1}{2\pi}\int_0^{2\pi} | h(re^{i\theta}) |^2 d\theta = \sum_{n=0}^{\infty} | a_n |^2 r^{2n},$$

we see that the hypothesis becomes $\sum_{n=0}^{\infty} | a_n |^2 \leqq 1$, while in a similar manner the desired conclusion becomes

$$\sum_{n=0}^{\infty} | c_n |^2 r^{2n} \leqq 1, \qquad \text{for } r \leqq p^{-1/2}.$$

Now $c_n = \sum a_{n_1} a_{n_2} \cdots a_{n_p}$, where the sum extends over all sets $(n_1, n_2, \cdots, n_p)$ of integers not less than 0 whose sum is n. Hence by the Cauchy-Schwarz inequality, we have

$$| c_n |^2 \leqq \sum 1 \cdot \sum | a_{n_1}^2 a_{n_2}^2 \cdots a_{n_p}^2 |,$$

where the sums have the same range as before. Now $\sum 1$ is the number of ways of distributing n units among p terms, and hence is not greater than p^n, which is the number of ways of distributing n *different* things among p sets. Hence for $r \leqq p^{-1/2}$ we have

$$| c_n |^2 r^{2n} \leqq \sum | a_{n_1}^2 a_{n_2}^2 \cdots a_{n_p}^2 |,$$

and therefore

$$\sum_{n=0}^{\infty} | c_n |^2 r^{2n} \leqq \left(\sum_{n=0}^{\infty} | a_n |^2 \right)^p \leqq 1.$$

The theorem is not true for $r > p^{-1/2}$. For if $\epsilon > 0$ and we put $f(z) = (1+\epsilon z)^2/(1+\epsilon^2)$, then the hypothesis of the theorem is satisfied. On the other hand, $f(z)^p = (1 + p\epsilon z + \cdots)^2/(1+p\epsilon^2 + \cdots)$, so that the mean of $|f(z)|^p$ on $|z| = r$ is $(1+p^2\epsilon^2 r^2 + \cdots)/(1+p\epsilon^2+\cdots)$, which is greater than 1 if $pr^2 > 1$ and ϵ is sufficiently small. This negative conclusion is true also for non-integral values of p; but we have been able to prove the positive statement only for integral values of p.

We turn now to the second paper mentioned. In this, it is proved that if $f(0) = 0$ and if the mean of $|f(z)|$ along each radius of the unit circle is not greater than 1, then the mean of $|f(z)|$ along $|z| = r$ is less than or equal to 1 for $r \leqq \frac{1}{2}$, but not always for a larger value of r. The negative part of the statement is immediate, the counter-example being $f(z) = 2z$. *We shall show that the hypothesis* $f(0) = 0$ *is unnecessary, and that the stronger statement that the mean of* $|f(z)|^2$ *along* $|z| = r$ *is less than or equal to* 1 *for* $r \leqq \frac{1}{2}$ *is also true.*

We prove first the following result. *If the mean of* $| F(z) |^2$ *on* $|z| = r$

is not greater than 1 *for* $r<1$, *then the mean of* $|F'(z)|^2$ *on* $|z|=r$ *is less than or equal to* 1 *for* $r\leqq\frac{1}{2}$. If we put $F(z)=\sum_{n=0}^{\infty}b_n z^n$, we have only to prove that

$$\sum_{n=1}^{\infty} n^2 |b_n|^2 r^{2(n-1)} \leqq \sum_{n=0}^{\infty} |b_n|^2 \qquad \text{for } r \leqq \tfrac{1}{2}.$$

Since $n\leqq 2^{n-1}$ for all positive integral values of n, we see that $nr^{n-1}\leqq 1$, so that the inequality is true. (The result is not correct for $r>\frac{1}{2}$; counterexample, $F(z)=z^2$.)

We suppose now that the mean of $|f(z)|$ along each radius of the unit circle is not greater than 1, and put

$$F(z) = \int_0^z f(\zeta)d\zeta.$$

Since the integral may be taken along a radius, we see that

$$|F(z)| \leqq 1, \qquad \text{for } |z| < 1.$$

Hence the mean of $|F(z)|^2$ on $|z|=r$ is certainly not greater than 1 for any $r<1$. Therefore the mean of $|F'(z)|^2=|f(z)|^2$ on $|z|=r$ is not greater than 1 for $r\leqq\frac{1}{2}$.

UNIVERSITY OF CALIFORNIA