THE CONFORMAL NEAR-MOEBIUS TRANSFORMATIONS[1]

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1. Introduction. In a previous paper,[2] we discussed the point transformations of the plane with reference to the maximum number of circles preserved. A nonconformal point transformation of the complex plane converts at most $2\infty^2$ circles into circles.[3] A conformal transformation, not of the Moebius type, carries at most $2\infty^1$ circles into circles (excluding the $2\infty^1$ minimal lines which become minimal lines). A Moebius transformation carries the entire family of ∞^3 circles into circles. From these results, we obtain the following two characterizations of the group of Moebius transformations: (1) if $3\infty^2$ circles are carried into circles after a point transformation, then the same is true for all circles, and the point transformation is therefore a Moebius transformation; and (2) any conformal transformation which converts $3\infty^1$ circles into circles is a Moebius transformation.

In this paper, we shall determine the set of all *conformal near-Moebius transformations*. That is, we shall obtain the set of all conformal transformations which convert *exactly* $2\infty^1$ circles into circles. *Any conformal near-Moebius transformation is of the form* M_2TM_1 *where* M_1 *and* M_2 *are Moebius transformations and* T *is any one of the three transformations* e^z, $\log z$, z^n. The two families preserved are two orthogonal pencils of circles.

The conformal near-collineation problem[4] is a special case of our problem. *Any conformal near-collineation is of the form* S_1TS_2 *where* S_1 *and* S_2 *are similitudes and* T *is any one of the three transformations* e^z, $\log z$, z^n. The family preserved is a pencil of straight lines (besides the $2\infty^1$ minimal lines).

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[1] Presented to the Society, February 24, 1940.

[2] Kasner and De Cicco, *Characterization of the Moebius group of circular transformations*, Proceedings of the National Academy of Sciences, vol. 25 (1939), pp. 209–213.

[3] In the previous paper, we derived these results for the point transformations of the *real* cartesian plane. But these same results may easily be derived for the complex cartesian plane without any difficulty. Note that a given family F of geometric configurations in the complex cartesian plane is said to possess ∞^n configurations if each of these is determined uniquely by n complex constants.

[4] Kasner, *The problem of partial geodesic representation*, Transactions of this Society, vol. 7 (1906), pp. 200–206. Also see Kasner, *The characterization of collineations*, this Bulletin, vol. 9 (1903), pp. 545–546; and Prenowitz, *The characterization of plane collineations in terms of homologous families of lines*, Transactions of this Society, vol. 38 (1935), pp. 564–599.

For our purposes, we shall find it convenient to define a point by the minimal coordinates (u, v) instead of the usual cartesian coordinates (x, y). The minimal and cartesian coordinates of the complex cartesian plane are connected by the two independent linear relations

$$u = x + iy, \qquad v = x - iy. \tag{1}$$

2. **The form of the differential equation of the two invariant families ($2\infty^1$) of circles.** In minimal coordinates, the ∞^3 circles (excluding the ∞^2 points and the $2\infty^1$ minimal lines) of the complex plane are represented by the ∞^3 hyperbolas which possess as asymptotes the minimal lines $u = \text{const.}$, and $v = \text{const.}$ Thus the family of ∞^3 circles is given by the equation

$$a_0uv + a_1u + a_2v + a_3 = 0, \tag{2}$$

where a_0, a_1, a_2, a_3 are complex constants such that either $a_0(a_1a_2 - a_3) \neq 0$ or $a_0 = 0$, $a_1a_2 \neq 0$. From this equation, it follows that the differential equation of the third order of the entire family of ∞^3 circles is

$$2pp'' = 3p'^2, \qquad p = dv/du. \tag{3}$$

To obtain the set of conformal (direct and reverse) near-Moebius transformations, it is only necessary to obtain the set of *direct* conformal near-Moebius transformations. For any reverse conformal near-Moebius transformation is the product of a direct conformal near-Moebius transformation by a reflection through the x axis (the straight line $u + v = 0$). Hence, in what follows, we shall only consider the set of direct conformal near-Moebius transformations.

In minimal coordinates, any direct conformal transformation is given by

$$U = \phi(u), \qquad V = \psi(v), \qquad \phi_u\psi_v \neq 0. \tag{4}$$

Upon extending this conformal transformation three times, we obtain

$$\begin{aligned}
P &= \frac{\psi_v}{\phi_u} p, \\
P' &= \frac{\psi_v}{\phi_u^2} p' + \frac{\psi_{vv}}{\phi_u^2} p^2 - \frac{\psi_v\phi_{uu}}{\phi_u^3} p, \\
P'' &= \frac{\psi_v}{\phi_u^3} p'' + \frac{3\psi_{vv}}{\phi_u^3} pp' - \frac{3\psi_v\phi_{uu}}{\phi_u^4} p' + \frac{\psi_{vvv}}{\phi_u^3} p^3 - \frac{3\psi_{vv}\phi_{uu}}{\phi_u^4} p^2 \\
&\quad + \left(- \frac{\psi_v\phi_{uuu}}{\phi_u^4} + \frac{3\psi_v\phi_{uu}^2}{\phi_u^5} \right) p.
\end{aligned} \tag{5}$$

For those circles which become circles under our conformal transformation, we know that the differential condition (3) must be preserved. Upon applying these conditions to our conformal transformation, we obtain the following theorem.

THEOREM 1. *The only possible circles which become circles under the conformal transformation* (4) (*not of the Moebius type*) *are the* $2\infty^1$ *circles whose differential equation is of the form*

$$p^2 = \frac{\psi_v^2(2\phi_u\phi_{uuu} - 3\phi_{uu}^2)}{\phi_u^2(2\psi_v\psi_{vvv} - 3\psi_{vv}^2)}. \tag{6}$$

We note that the two families ($2\infty^1$) of circles (if they exist) are orthogonal.

3. **The $2\infty^1$ circles whose differential equation is of the form** (6). The two families of circles are given by a differential equation of the form

$$p = \pm\frac{\gamma(u)}{\delta(v)}. \tag{7}$$

We shall find all families of circles whose differential equation is of this form. By means of the Moebius transformations, we shall reduce our results to canonical forms.

For the time being we shall consider only the plus sign. We observe first that *neither γ nor δ can be zero*. For then the circles would be the minimal lines u or v = const. These are excluded from consideration.

Upon taking the first and second derivatives of (7), we obtain

$$\begin{aligned} p' &= \frac{\gamma_u}{\delta} - \frac{\gamma^2\delta_v}{\delta^3}, \\ p'' &= \frac{\gamma_{uu}}{\delta} - \frac{3\gamma\gamma_u\delta_v}{\delta^3} - \frac{\gamma^3\delta_{vv}}{\delta^4} + \frac{3\gamma^3\delta_v^2}{\delta^5}. \end{aligned} \tag{8}$$

Substituting (7) and (8) into (3), we find that (7) *represents the differential equation of* $2\infty^1$ *circles if and only if*

$$\frac{2\gamma\gamma_{uu} - 3\gamma_u^2}{\gamma^4} = \frac{2\delta\delta_{vv} - 3\delta_v^2}{\delta^4}. \tag{9}$$

This equation will be true if and only if each side is equal to the same complex constant a^2. Hence

$$(10) \qquad 2\gamma\gamma_{uu} - 3\gamma_u^2 = a^2\gamma^4, \qquad 2\delta\delta_{vv} - 3\delta_v^2 = a^2\delta^4.$$

We proceed to discuss the solution of these two differential equations. We shall divide this discussion into several parts.

Case A. *Let* $a=0$. *In this case, the equation* (7) *represents two orthogonal parabolic pencils of circles* (*or their Moebius equivalents*). By a Moebius transformation, these may be reduced to the two orthogonal parallel pencils of straight lines

$$(\mathrm{A}') \qquad v = \pm u + \text{const.}$$

Hence the canonical form of the differential equation (7) for Case A is

$$(\mathrm{A}) \qquad p^2 = 1.$$

(I) First let neither γ nor δ be constants. Then (10) may be written in the form

$$(11) \qquad 2\frac{d}{du}\left(\frac{\gamma_u}{\gamma}\right) = \left(\frac{\gamma_u}{\gamma}\right)^2, \qquad 2\frac{d}{dv}\left(\frac{\delta_v}{\delta}\right) = \left(\frac{\delta_v}{\delta}\right)^2.$$

The integration of these yields

$$(12) \qquad \gamma = \frac{b}{(u-u_0)^2}, \qquad \delta = \frac{c}{(v-v_0)^2},$$

where u_0, v_0, b, c are constants. Substituting these into (7) and integrating the resulting differential equation, we find

$$(13) \qquad \frac{c}{v-v_0} = \pm\frac{b}{u-u_0} + \text{const.}$$

These are two orthogonal parabolic pencils of circles. By an appropriate Moebius transformation, we can convert (13) into (A$'$). Hence the canonical form of our differential equation (7) for this case is (A).

(II) Next let γ and δ be constants. By integrating (7), we find that the two families are two orthogonal parallel pencils of straight lines (which are equivalent by a Moebius transformation to two orthogonal parabolic pencils of circles). By a similitude, we can carry these into (A$'$). Hence the canonical form of (7) in this case is also equation (A).

(III) Let γ be not constant and δ constant. We find that γ and δ are given by

$$(14) \qquad \gamma = \frac{-b}{(u-u_0)^2}, \qquad \delta = \text{const.},$$

where u_0 and b are constants. Substituting these into (7) and integrating, we obtain

$$\delta v = \pm \frac{b}{u - u_0} + \text{const.} \tag{15}$$

The first family consists of all circles of the same radius $r=(b/\delta)^{1/2}$ and with centers on the minimal line $u=u_0$. The orthogonal family consists of all circles with centers on the minimal line $u=u_0$ and of radius ir. These two families of circles have the same radical axis, namely, their common line of centers $u=u_0$. Also these circles are tangent (and orthogonal) to each other at the point at infinity on this minimal line $u=u_0$.

These two families are equivalent by a Moebius transformation to two orthogonal parabolic pencils of circles. By an appropriate Moebius transformation, these two families (15) can be carried into the two orthogonal parallel pencils of straight lines (A′). Hence the canonical form of (7) in this case is also (A).

(IV) Finally let γ be constant and δ not constant. Then γ and δ are given by

$$\gamma = \text{const.}, \qquad \delta = -\frac{c}{(v - v_0)^2}\cdot \tag{16}$$

Substituting these into (7) and integrating the resulting differential equation, we find

$$\frac{c}{v - v_0} = \pm\, \gamma u + \text{const.}, \tag{17}$$

where v_0 and c are constants.

The first family consists of all circles of the same radius $r=(c/\gamma)^{1/2}$ and with centers on the minimal line $v=v_0$. The orthogonal family consists of all circles with centers on the minimal line $v=v_0$ and of radius ir. The two families of circles have the same radical axis, namely, their common line of centers $v=v_0$. Also these circles are tangent (and orthogonal) to each other at the point at infinity on this minimal line $v=v_0$.

These two families are equivalent by a Moebius transformation to two orthogonal parabolic pencils of circles. By an appropriate Moebius transformation, these two families (17) can be converted into (A′). Thus the canonical form of (7) in this case is again (A).

Case B. *Let* $a\neq 0$. *In this case, the differential equation* (7) *represents two orthogonal nonparabolic pencils of circles* (*or their Moebius equiva-*

lents). By a Moebius transformation, these may be reduced to the pencil of straight lines with vertex at origin and to the pencil of concentric circles with their common center at the origin

$$v/u = \text{const.}, \qquad uv = \text{const.} \tag{B'}$$

Hence the canonical form of the differential equation (7) for this case is

$$p^2 = v^2/u^2. \tag{B}$$

In this case, γ (and δ) cannot be constant. Let $\lambda = \gamma\gamma_u$. The first of equations (10) can be written in the form

$$2\gamma\lambda\lambda_\gamma - 5\lambda^2 = a^2\gamma^6. \tag{18}$$

Multiplying this by γ^{-6} and integrating, we find

$$\lambda^2 = \gamma^5(a^2\gamma + 2b), \tag{19}$$

where a and b are constants. Replacing λ by $\gamma\gamma_u$, this equation (and a similar equation for δ) can be written in the form

$$\gamma_u = \gamma^2(a^2 + 2b/\gamma)^{1/2}, \qquad \delta_v = \delta^2(a^2 + 2c/\delta)^{1/2}, \tag{20}$$

where a, b, c are constants.

(I) Let neither b nor c be zero. The preceding equations then yield

$$\gamma = \frac{2b}{b^2(u - u_0)^2 - a^2}, \qquad \delta = \frac{2c}{c^2(v - v_0)^2 - a^2}. \tag{21}$$

Substituting these into (7) and integrating, we obtain the two families of circles

$$\begin{aligned} \frac{c(v - v_0) - a}{c(v - v_0) + a} &= \text{const.} \frac{b(u - u_0) - a}{b(u - u_0) + a}, \\ \frac{c(v - v_0) - a}{c(v - v_0) + a} &= \text{const.} \frac{b(u - u_0) + a}{b(u - u_0) - a}. \end{aligned} \tag{22}$$

These two families represent two orthogonal nonparabolic pencils of circles. The circles of the first family are those passing through the two points $(u_0+a/b, v_0+a/c)$ and $(u_0-a/b, v_0-a/c)$. The circles of the second family are those passing through the two points $(u_0+a/b, v_0-a/c)$ and $(u_0-a/b, v_0+a/c)$.

The two orthogonal nonparabolic pencils of circles (22) have the same center (u_0, v_0). This is the intersection of the two orthogonal radical axes and of the two orthogonal lines of centers. If d is the distance between the two fixed points of the first pencil of circles, then id is the distance between the two fixed points of the second pencil. These properties are characteristic for two orthogonal nonparabolic pencils of circles.

By an appropriate Moebius transformation, the two families (22) can be converted into (B′). Thus the canonical form of the differential equation (7) for this case is (B).

(II) Let $b=c=0$. From (20), we find

$$\gamma = \frac{1}{a(u-u_0)}, \qquad \delta = \frac{1}{a(v-v_0)}\cdot \tag{23}$$

Substituting these into (7) and integrating, we obtain the two families of circles

$$\frac{v-v_0}{u-u_0} = \text{const.}, \qquad (u-u_0)(v-v_0) = \text{const.} \tag{24}$$

The first family is a pencil of straight lines with vertex at (u_0, v_0). The second family is a pencil of concentric circles with the common center at (u_0, v_0). By an appropriate translation, these two families can be carried into (B′). Thus, in this case, the canonical form of the differential equation (7) is (B).

(III) Let $b=0$ and $c\neq 0$. In this case, we have

$$\gamma = \frac{1}{a(u-u_0)}, \qquad \delta = \frac{2c}{c^2(v-v_0)^2 - a^2}\cdot \tag{25}$$

Substituting these into (7) and integrating, we find the two orthogonal pencils of circles

$$\begin{aligned} \frac{c(v-v_0)-a}{c(v-v_0)+a} &= \text{const.}\ (u-u_0), \\ \frac{c(v-v_0)-a}{c(v-v_0)+a} &= \text{const.}\ \frac{1}{u-u_0}\cdot \end{aligned} \tag{26}$$

The circles of the first family are those passing through the point $(u_0, v_0+a/c)$ and the point at infinity on the minimal line $v=v_0-a/c$. The circles of the second family are those passing through the point $(u_0, v_0-a/c)$ and the point at infinity on the minimal line $v=v_0+a/c$.

The radical axis and the line of centers (or the line of centers and the radical axis) of the first (or second) family of circles are respectively the minimal lines $v=v_0+a/c$ and $v=v_0-a/c$.

Under the group of Moebius transformations, these two families are equivalent to any two orthogonal nonparabolic pencils of circles. By an appropriate Moebius transformation, these two families can be converted into (B′). Thus the canonical form of the differential equation (7) for this case is (B).

(IV) Let $b \neq 0$ and $c=0$. The two families of circles are then

$$(27)\quad v - v_0 = \text{const.}\,\frac{b(u - u_0) - a}{b(u - u_0) + a}, \quad v - v_0 = \text{const.}\,\frac{b(u - u_0) + a}{b(u - u_0) - a}.$$

The circles of the first family are those passing through the point $(u_0 + a/b, v_0)$ and the point at infinity on the minimal line $u = u_0 - a/b$. The circles of the second family are those passing through the point $(u_0 - a/b, v_0)$ and the point at infinity on the minimal line $u = u_0 + a/b$. The radical axis and the line of centers (or the line of centers and the radical axis) of the first (or the second) family of circles are respectively the minimal lines $u = u_0 + a/b$ and $u = u_0 - a/b$.

Under the group of Moebius transformations, these two families are equivalent to any two orthogonal nonparabolic pencils of circles. By an appropriate Moebius transformation, these two families can be converted into (B′). Thus the canonical form of the differential equation (7) for this case is (B).

Thus in all cases, we have proved the following theorem:

THEOREM 2. *The only* $2\infty^1$ *circles whose differential equation is of the form* (7) *are two orthogonal pencils of circles. Under the group of Moebius transformations, these may be classified into two distinct types*: (1) *two orthogonal parabolic pencils of circles and* (2) *two orthogonal nonparabolic pencils of circles.*

The canonical forms of the differential equation (7) of types (1) and (2) are respectively (A) and (B).

4. **The conformal near-Moebius transformations.** We now proceed to find the conformal near-Moebius transformations. First it is seen that *the inverse* N^{-1} *of any conformal near-Moebius transformation* N *is also a conformal near-Moebius transformation.* For since N is a one-to-one correspondence which carries $2\infty^1$ circles into circles, it follows that the transformed circles must be $2\infty^1$ in number. Hence the inverse N^{-1} carries $2\infty^1$ circles into circles, and therefore it must be a conformal near-Moebius transformation.

Let N be any conformal near-Moebius transformation. Then there exist exactly $2\infty^1$ circles which are preserved by N. Denote these circles by γ and their transformed circles under N by Γ. That is,

$N(\gamma)=\Gamma$, and $\gamma=N^{-1}(\Gamma)$. By Theorems 1 and 2, we find that γ *and* Γ *are each two orthogonal pencils of circles.*

Let γ_0 denote the canonical form (A′) and (A), or (B′) and (B) of two orthogonal parabolic, or nonparabolic, pencils of circles. Let M_1 by any Moebius transformation which carries the two orthogonal pencils of circles γ into the canonical form γ_0, and let M_2 be any Moebius transformation which converts the canonical form γ_0 into the two orthogonal pencils of circles Γ. Then the transformation $T=M_2^{-1}NM_1^{-1}$ preserves the canonical form γ_0 of two orthogonal pencils of circles. Hence *any conformal near-Moebius transformation N is of the form M_2TM_1 where M_1 and M_2 are Moebius transformations and T is any conformal near-Moebius transformation which preserves the canonical form γ_0 of two orthogonal pencils of circles.*

Next we shall show that any transformation T which preserves the canonical form γ_0 of two orthogonal pencils of circles must be one of the three types: (1) $U=be^{au}$, $V=ce^{\pm av}$; (2) $U=a\log u+b$, $V=\pm a\log v+c$; and (3) $U=bu^n$, $V=cv^{\pm n}$. Obviously T converts either (I) (A) into (A), or (II) (A) into (B), or (III) (B) into (A), or (IV) (B) into (B).

(I) Let T convert (A) into (A). The differential equation $p^2=1$ must be preserved. For this to be so, we find that T must be of the form $U=au+b$, $V=\pm av+c$ where $a\neq 0$, b, c are constants. This of course is a simple case of the Moebius transformations. Hence we conclude that *there are no conformal near-Moebius transformations which preserve two orthogonal parabolic pencils of circles.*

(II) Let T convert (A) into (B). The differential equation $p^2=1$ is converted into $P^2=V^2/U^2$. For this to be so, we find that T must be of the form $U=be^{au}$, $V=ce^{\pm av}$, where a, b, c are nonzero constants. Hence we conclude that *any conformal near-Moebius transformation N which carries two orthogonal parabolic pencils of circles into two orthogonal nonparabolic pencils of circles must be of the form M_2TM_1 where M_1 and M_2 are Moebius transformations and T is the transformation $U=e^u$, $V=e^v$.*

(III) Let T convert (B) into (A). The differential equation $p^2=v^2/u^2$ is converted into $P^2=1$. Hence T must be of the form $U=a\log u+b$, $V=\pm a\log v+c$, where $a\neq 0$, b, c are constants. Thus we conclude that *any conformal near-Moebius transformation N which carries two orthogonal nonparabolic pencils of circles into two orthogonal parabolic pencils must be of the form M_2TM_1 where M_1 and M_2 are Moebius transformations and T is the transformation $U=\log u$, $V=\log v$.*

(IV) Let T convert (B) into (B). The differential equation $p^2=v^2/u^2$ must be carried into the differential equation $P^2=V^2/U^2$.

Hence T must be of the form $U=bu^n$, $V=cv^{\pm n}$ where n, b, c are nonzero constants. Thus *any conformal near-Moebius transformation N which carries two orthogonal nonparabolic pencils of circles into two orthogonal nonparabolic pencils of circles must be of the form M_2TM_1 where M_1 and M_2 are Moebius transformations and T is the transformation $U=u^n$, $V=v^n$ with n a complex nonzero constant.*

We therefore obtain from the preceding results the following:

THEOREM 3. *Any conformal near-Moebius transformation N of the complex cartesian plane is of the form M_2TM_1 where M_1 and M_2 are Moebius transformations and T is any of the three transformations* (1) $U=e^u$, $V=e^v$; (2) $U=\log u$, $V=\log v$; and (3) $U=u^n$, $V=v^n$ (*n a complex nonzero constant*).

In this paper, we have given all the conformal near-Moebius transformations, that is, all the conformal transformatons which preserve *exactly* $2\infty^1$ circles. In a later paper, we shall give the set of all nonconformal near-Moebius transformations, that is, the set of all nonconformal transformations which preserve *exactly* $2\infty^2$ circles.

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