ON TOPOLOGICAL COMPLETENESS[1]

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Recently A. Weil[2] defined a uniform space as a set of points p such that for each α in a set A there is defined a set $U_\alpha(p) \subset S$, the class of sets $U_\alpha(p)$ satisfying the conditions:

$\mathrm{I_A}$. $\prod_\alpha U_\alpha(p) = (p)$.

$\mathrm{II_A}$. To each α, $\beta \in A$ there is a $\gamma = \gamma(\alpha, \beta) \in A$ such that $U_\gamma(p) \subset U_\alpha(p) U_\beta(p)$.

$\mathrm{III_A}$. To each $\alpha \in A$ there is a $\beta(\alpha) \in A$ such that if p', $p'' \in U_{\beta(\alpha)}(q)$, then $p'' \in U_\alpha(p')$.

For the uniform space S, Weil introduced the concept of Cauchy family $\{M_\beta\}$ of sets. Such a family is defined by the conditions that the intersection of any finite number of sets of the family is non-empty and that to each $\alpha \in A$ there is a $p_\alpha \in S$ and a $\beta(\alpha)$ such that $M_{\beta(\alpha)} \subset U_\alpha(p_\alpha)$. Weil gives a theory of completeness in these terms.

The writer has considered[3] a space S of points p and neighborhoods $U_\alpha(p)$ where α is an element of a set A such that:

I. $\prod_\alpha U_\alpha(p) = (p)$.

II. To each α and $\beta \in A$ and $p \in S$ there is a $\gamma = \gamma(\alpha, \beta; p)$ such that $U_\gamma(p) \subset U_\alpha(p) U_\beta(p)$.

III. To each $\alpha \in A$ and $p \in S$ there are $\lambda(\alpha)$, $\delta(p, \alpha) \in A$ such that, if $U_{\delta(p,\alpha)}(q) U_{\lambda(\alpha)}(p) \neq 0$, then $U_{\delta(p,\alpha)}(q) \subset U_\alpha(p)$.

The uniformity conditions here are lighter than those in $\mathrm{II_A}$ and $\mathrm{III_A}$. A Cauchy sequence $p_n \in S$ was defined by the condition that for every $\alpha \in A$, n_α and $p_\alpha \in S$ exist such that $p_n \in U_\alpha(p_\alpha)$ for $n \geqq n_\alpha$. S is complete if every Cauchy sequence has a limit. It was shown that there is a complete space S^* which contains a homeomorphic image of S such that the image of a Cauchy sequence in S is a convergent sequence in S^*.

It is the object of this paper to show that Weil's space is a special case of the space $S_{\mathrm{I,II,III}}$ and that the notion of Cauchy family in this space leads to the same theory of completeness as that previously developed.

THEOREM 1. *If S satisfies* $\mathrm{III_A}$ *and* $\beta^2(\alpha) = \beta(\beta(\alpha))$, *then from* $U_{\beta^2(\alpha)}(q) U_{\beta^2(\alpha)}(p) \neq 0$ *follows* $U_{\beta^2(\alpha)}(q) \subset U_\alpha(p)$.

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[1] Presented to the Society, December 27, 1939.

[2] A. Weil, *Sur les Espaces à Structure Uniforme*, Paris, 1938.

[3] L. W. Cohen, *On imbedding a space in a complete space*, Duke Mathematical Journal, vol. 5 (1939), pp. 174–183. Also Duke Mathematical Journal, vol. 3 (1937), pp. 610–617, where the notion of topological completeness is introduced.

PROOF. Let $s \in U_{\beta^2(\alpha)}(q) U_{\beta^2(\alpha)}(p)$. Then from $s, q' \in U_{\beta^2(\alpha)}(q)$ we have $q' \in U_{\beta(\alpha)}(s)$. Therefore $U_{\beta^2(\alpha)}(q) \subset U_{\beta(\alpha)}(s)$. Similarly $U_{\beta^2(\alpha)}(p) \subset U_{\beta(\alpha)}(s)$. Now from $p \in U_{\beta(\alpha)}(s)$ and $U_{\beta^2(\alpha)}(q) \subset U_{\beta(\alpha)}(s)$ we have $U_{\beta^2(\alpha)}(q) \subset U_\alpha(p)$.

COROLLARY. *If S satisfies* III_A, *then S satisfies* III.

PROOF. For any $p \in S$ and $\alpha \in A$ we need only take $\lambda(\alpha) = \delta(p, \alpha) = \beta(\beta(\alpha))$. The result is stronger than III since $\delta(p, \alpha)$ is independent of p.

From now on a space S is one satisfying I, II, III. A family of sets $\{M_\beta\}$ is a Cauchy family if the intersection of any finite number of M_β is non-empty and if for any $\alpha \in A$ there is a $\beta(\alpha)$ such that $M_{\beta(\alpha)} \subset U_\alpha(p_\alpha)$ for some $p_\alpha \in S$. We will say that S is W-complete if, for every Cauchy family $\{M_\beta\}$, $\prod_\beta \overline{M}_\beta \neq 0$, where $\overline{M}_\beta$ is the closure of M_β. We shall always use the notations $\lambda(\alpha)$, $\delta(p, \alpha)$ in the sense of III.

THEOREM 2. *S is W-complete if and only if every Cauchy family $\{U_\alpha(p_\alpha)\}$ consisting of one $U_\alpha(p_\alpha)$ for each $\alpha \in A$ has the property that* $\prod_\alpha U_\alpha(p_{\lambda(\alpha)}) \neq 0$.

PROOF. Assume S is W-complete. If $\{U_\alpha(p_\alpha)\}$ is a Cauchy family, then $\prod_\alpha \overline{U}_{\lambda(\alpha)}(p_{\lambda(\alpha)}) \neq 0$. Since $\overline{U}_{\lambda(\alpha)}(p) \subset U_\alpha(p)$ follows from III, the condition $\prod_\alpha U_\alpha(p_{\lambda(\alpha)}) \neq 0$ is necessary. Assume now that the condition is satisfied and let $\{M_\beta\}$ be a Cauchy family in S. To each $\alpha \in A$ there are $p_\alpha \in S$ and $M_{\beta(\alpha)}$ such that $M_{\beta(\alpha)} \subset U_\alpha(p_\alpha)$. It is clear that $\{U_\alpha(p_\alpha)\}$ is a Cauchy family. Hence there is $p \in \prod_\alpha U_\alpha(p_{\lambda(\alpha)})$. For any $\gamma \in A$, consider the $\lambda(\gamma)$ and $\alpha = \delta(p, \gamma)$ of III. Since $p \in U_\alpha(p_{\lambda(\alpha)}) U_{\lambda(\gamma)}(p)$, $U_\alpha(p_{\lambda(\alpha)}) \subset U_\gamma(p)$ and

$$M_{\beta(\lambda(\alpha))} \subset U_{\lambda(\alpha)}(p_{\lambda(\alpha)}) \subset U_\alpha(p_{\lambda(\alpha)}) \subset U_\gamma(p).$$

Hence

$$0 \neq M_\beta M_{\beta(\lambda(\alpha))} \subset M_\beta U_\gamma(p)$$

for all β, γ. Thus $p \in \prod_\beta \overline{M}_\beta$ and S is W-complete.

The space S^* referred to above is defined as follows. A family $\{U_\alpha(p_\alpha)\}$, one for each $\alpha \in A$, such that for any $(\alpha_1, \cdots, \alpha_n) \subset A$, $U_{\alpha_1}(p_{\alpha_1}) U_{\alpha_2}(p_{\alpha_2}) \cdots U_{\alpha_n}(p_{\alpha_n}) \neq 0$ is denoted by Π. We write $\Pi' \sim \Pi''$ if for every $\alpha \in A$ there is a set $(\alpha_1, \cdots, \alpha_n) \subset A$ such that for some $p_\alpha \in S$

$$\prod_{i=1}^{n} U_{\alpha_i}(p'_{\alpha_i}) + \prod_{i=1}^{n} U_{\alpha_i}(p''_{\alpha_i}) \subset U_\alpha(p_\alpha).$$

The relation $\Pi' \sim \Pi''$ classifies all Π into mutually exclusive classes $C(\Pi)$ which are the points P of S^*. The neighborhoods $U^*_{\alpha_1 \cdots \alpha_n}(P)$ are made up of all $Q = C(\Pi^Q) \varepsilon S^*$ where $\Pi^Q = \{U_\alpha(q_\alpha)\}$ is such that for some $\beta_i = \beta_i(\alpha_1, \cdots, \alpha_n; P)$, $i = 1, \cdots, m = m(\alpha_1, \cdots, \alpha_n; P)$,

$$\prod_{i=1}^{m} U_{\beta_i}(q_{\beta_i}) \subset \prod_{i=1}^{n} U_{\alpha_i}(p_{\alpha_i}).$$

The space S^* (for which A^* is the set of all finite subsets of A) satisfies I, II and

III*. For each $(\alpha_1, \cdots, \alpha_n) \subset A$ and for $P \varepsilon S^*$ there exist $\gamma_i = \gamma_i(\alpha_1, \cdots, \alpha_n; P)$, $i = 1, \cdots, m = m(\alpha_1, \cdots, \alpha_n; P)$, such that if $U^*_{\gamma_1 \cdots \gamma_m}(Q) U^*_{\gamma_1 \cdots \gamma_m}(P) \neq 0$, then $U^*_{\gamma_1 \cdots \gamma_m}(Q) \subset U^*_{\alpha_1 \cdots \alpha_n}(P)$.

The mapping $f(p) = C(\Pi^p)$, where $\Pi^p = \{U_\alpha(p)\}$, is the homeomorphism on S to a subset of S^* referred to above.

THEOREM 3. *S^* is W-complete.*

PROOF. We first show that if $\{U^*_{\alpha_1 \cdots \alpha_n}(P^{\alpha_1 \cdots \alpha_n})\}$, where each $(\alpha_1, \cdots, \alpha_n) \subset A$ occurs just once, is a Cauchy family in S^*, then

$$\prod_{(\alpha_1 \cdots \alpha_n) \subset A} \overline{U^*_{\alpha_1 \cdots \alpha_n}}(p^{\alpha_1 \cdots \alpha_n}) \neq 0.$$

Consider the $U^*_\alpha(P^\alpha)$ for all $\alpha \varepsilon A$. Now $P^\alpha = C(\Pi^{P^\alpha})$, $\Pi^{P^\alpha} = \{U_\gamma(p^\alpha_\gamma)\}$, $U_\gamma(p^\alpha_\gamma)$ being a neighborhood in S for each $(\alpha, \gamma) \subset A$. Since, for each $(\alpha_1, \cdots, \alpha_n) \subset A$, $\prod_{i=1}^n U^*_{\alpha_i}(P^{\alpha_i}) \neq 0$, we have

$$\prod_{i=1}^{n} U_{\alpha_i}(p^{\alpha_i}_{\alpha_i}) \neq 0.$$

Hence the family $\Pi = \{U_\alpha(p^\alpha_\alpha)\}$ defines a $P = C(\Pi) \varepsilon S^*$. For any $(\gamma_1, \cdots, \gamma_m) \subset A$

$$\prod_{i=1}^{m} U^*_{\gamma_i}(P) = U^*_{\gamma_1 \cdots \gamma_m}(P).$$

Both $U^*_{\gamma_i}(P)$ and $U^*_{\gamma_i}(P^{\gamma_i})$ are the set of all $Q = C(\Pi^Q)$, $\Pi^Q = \{U_\beta(q_\beta)\}$ such that for some $\beta_{ji} = \beta_j(\gamma_i)$, $j = 1, \cdots, k_i$, $\prod_{j=1}^{k_i} U_{\beta_{ji}}(q_{\beta_{ji}}) \subset U_{\gamma_i}(p^{\gamma_i}_{\gamma_i})$. Hence

$$U^*_{\gamma_i}(P) = U^*_{\gamma_i}(P^{\gamma_i}), \qquad U^*_{\gamma_1 \cdots \gamma_m}(P) = \prod_{i=1}^{m} U^*_{\gamma_i}(P^{\gamma_i}).$$

Thus from the fact that $\{U^*_{\alpha_1 \cdots \alpha_n}(P^{\alpha_1 \cdots \alpha_n})\}$ is a Cauchy family we have for any sets $(\alpha_1, \cdots, \alpha_n)$, $(\gamma_1, \cdots, \gamma_m) \subset A$

$$U^*_{\alpha_1\cdots\alpha_n}(P^{\alpha_1\cdots\alpha_n})U^*_{\gamma_1\cdots\gamma_m}(P) = U^*_{\alpha_1\cdots\alpha_n}(P^{\alpha_1\cdots\alpha_n})\prod_{i=1}^{m} U^*_{\gamma_i}(P^{\gamma_i}) \neq 0.$$

It follows that

$$(1) \qquad P \,\varepsilon \prod_{(\alpha_1\cdots\alpha_n)\subset A} \overline{U^*_{\alpha_1\cdots\alpha_n}}(P^{\alpha_1\cdots\alpha_n}).$$

Now suppose that $\{M_\beta^*\}$ is a Cauchy family of sets in S^*. For every $(\gamma_1, \cdots, \gamma_n) \subset A$ there is a $\beta(\gamma_1, \cdots, \gamma_n)$ such that for some $P^{\gamma_1\cdots\gamma_n} \,\varepsilon\, S^*$, $M^*_{\beta(\gamma_1\cdots\gamma_n)} \subset U^*_{\gamma_1\cdots\gamma_n}(P^{\gamma_1\cdots\gamma_n})$. Thus

$$0 \neq \prod_{i=1}^{m} M^*_{\beta(\gamma_1^i,\cdots,\gamma_{n_i}^i)} \subset \prod_{i=1}^{m} U^{*i}_{\gamma_1^i,\cdots,\gamma_{n_i}^i}(P^{\gamma_1^i,\cdots,\gamma_{n_i}^i})$$

for every finite set $(\gamma_1^i, \cdots, \gamma_{n_i}^i) \subset A$ and $\{U^*_{\gamma_1\cdots\gamma_n}(P^{\gamma_1\cdots\gamma_n})\}$ is a Cauchy family. Let $P \,\varepsilon\, S^*$ satisfy (1). For any $(\alpha_1, \cdots, \alpha_n) \subset A$ and $P \,\varepsilon\, S^*$ let $(\gamma_1, \cdots, \gamma_m) \subset A$ be the set of $\gamma_i = \gamma_i(\alpha_1, \cdots, \alpha_n; P)$ satisfying III*. From III* and (1)

$$U^*_{\gamma_1\cdots\gamma_m}(P^{\gamma_1\cdots\gamma_m}) \subset U^*_{\alpha_1\cdots\alpha_n}(P).$$

Since $\{M_\beta^*\}$ is a Cauchy family, for any β and $(\alpha_1, \cdots, \alpha_n) \subset A$ we have

$$0 \neq M^*_\beta M^*_{\beta(\gamma_1,\cdots,\gamma_m)} \subset M^*_{\beta(\gamma_1,\cdots,\gamma_m)} \subset U^*_{\gamma_1\cdots\gamma_m}(P^{\gamma_1\cdots\gamma_m}) \subset U^*_{\alpha_1\cdots\alpha_n}(P),$$

$$M^*_\beta U^*_{\alpha_1\cdots\alpha_n}(P) \neq 0.$$

Hence $P \,\varepsilon \prod_\beta \overline{M_\beta^*}$ and S^* is W-complete.

THEOREM 4. *If $\{M_\beta\}$ is a Cauchy family in S and $f(S) \subset S^*$ is the homeomorphism defined above, then there is a $P \,\varepsilon\, S^*$ such that*

1. $\prod_\beta \overline{f(M_\beta)} = (P)$,
2. *for any $(\alpha_1, \cdots, \alpha_n) \subset A$ there are $\beta_1, \cdots, \beta_m$ such that*

$$\prod_{i=1}^{m} f(M_{\beta_i}) \subset U^*_{\alpha_1\cdots\alpha_n}(P).$$

PROOF. $f(M_\beta)$ is the class of $P = C(\Pi^p)$ for all $p \,\varepsilon\, M_\beta$ where for any $(\alpha_1, \cdots, \alpha_n) \subset A$ and $\lambda(\alpha_i)$ there are $\beta(\lambda(\alpha_i))$ and $p_{\lambda(\alpha_i)}$ such that

$$M_{\beta(\lambda(\alpha_i))} \subset U_{\lambda(\alpha_i)}(p_{\lambda(\alpha_i)}) \subset U_{\alpha_i}(p_{\lambda(\alpha_i)}),$$

$$0 \neq \prod_{i=1}^{n} M_{\beta(\lambda(\alpha_i))} \subset \prod_{i=1}^{n} U_{\alpha_i}(p_{\lambda(\alpha_i)}).$$

If $q \,\varepsilon \prod_{i=1}^n M_{\beta(\lambda(\alpha_i))}$, then for $\delta_i = \delta(p_{\lambda(\alpha_i)}, \alpha_i)$ we have

$$q \,\varepsilon\, U_{\delta_i}(q)U_{\lambda(\alpha_i)}(p_{\lambda(\alpha_i)}), \qquad U_{\delta_i}(q) \subset U_{\alpha_i}(p_{\lambda(\alpha_i)}), \qquad i = 1, 2, \cdots, n.$$

Since

$$q \,\varepsilon \prod_{i=1}^{n} U_{\delta_i}(q) \subset \prod_{i=1}^{n} U_{\alpha_i}(p_{\lambda(\alpha_i)}),$$

the family $\{U_\alpha(p_{\lambda(\alpha)})\} = \Pi$ defines a $P = C(\Pi) \,\varepsilon\, S^*$ such that

$$f(q) \,\varepsilon\, U^*_{\delta_1\cdots\delta_n}(f(q)) \subset U^*_{\alpha_1\cdots\alpha_n}(P).$$

Hence for such $(\alpha_1, \cdots, \alpha_n) \subset A$ there are $P^{\alpha_1\cdots\alpha_n} = P$ and $\beta_i = \beta(\lambda(\alpha_i))$, $i = 1, \cdots, n$, such that

$$0 \neq \prod_{i=1}^{n} f(M_{\beta(\lambda(\alpha_i))}) \subset U^*_{\alpha_1\cdots\alpha_n}(P).$$

Thus the family of all finite products $\{f(M_{\beta_1}) \cdots f(M_{\beta_n})\}$ is a Cauchy family in S^* since $f(M_{\beta_1}) \cdots f(M_{\beta_n}) = f(M_{\beta_1} \cdots M_{\beta_n}) \neq 0$. We have

$$U^*_{\alpha_1\cdots\alpha_n}(P) f(M_\beta) \supset \prod_{i=1}^{n} f(M_{\beta(\lambda(\alpha_i))}) f(M_\beta) \neq 0$$

for all β and all $(\alpha_1, \cdots, \alpha_n) \subset A$. In other words

$$P \,\varepsilon \prod_{\beta} \overline{f(M_\beta)}.$$

But the intersection of all $\overline{f(M_\beta)}$ is P, for if P' is any other point there are sets $(\alpha_1, \cdots, \alpha_n)$, $(\alpha_1', \cdots, \alpha_{n'}') \subset A$ such that

$$U^*_{\alpha_1\cdots\alpha_n}(P) U^*_{\alpha_1'\cdots\alpha_{n'}'}(P') = 0$$

and

$$P' \,\varepsilon\, S^* - \overline{U^*_{\alpha_1\cdots\alpha_n}(P)} \subset S^* - \prod_{i=1}^{n} \overline{f(M_{\beta(\lambda(\alpha_i))})} \subset S^* - \prod_{\beta} \overline{f(M_\beta)}.$$

We conclude with the remark that if S is W-complete, S is complete. Suppose p_n is a Cauchy sequence in S. Let $M_n = (p_n, p_{n+1}, \cdots)$. Then the intersection of any finite set of M_n is non-empty and for any $\alpha \,\varepsilon\, A$ there is an $n(\alpha)$ such that $M_{n(\alpha)} \subset U_\alpha(p_\alpha)$ for some $p_\alpha \,\varepsilon\, S$. Thus $\{M_n\}$ is a Cauchy family. S being W-complete, there is a $p \,\varepsilon \prod_n \overline{M}_n$. Now for p, any $\alpha \,\varepsilon\, A$, $\lambda(\alpha)$ and $\delta = \delta(p, \alpha)$, we have

$$M_{n(\delta)} \subset U_\delta(p_\delta), \qquad 0 \neq U_{\lambda(\alpha)}(p) M_{n(\delta)} \subset U_{\lambda(\alpha)}(p) U_\delta(p_\delta),$$

$$M_{n(\delta)} \subset U_\delta(p_\delta) \subset U_\alpha(p), \quad p_n \,\varepsilon\, U_\alpha(p), \quad n \geqq n(\delta) = n(\delta(p, \alpha)), \lim p_n = p.$$

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