ON THE LOGARITHMIC SOLUTIONS OF THE GENERALIZED HYPERGEOMETRIC EQUATION WHEN $p=q+1$

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1. **Introduction.** In a recent paper,* the author gave the relations among the non-logarithmic solutions of the equation

$$\left\{ \prod_{t=1}^{q+1} (\theta + a_t) - \frac{1}{z} \prod_{t=1}^{q+1} (\theta + c_t - 1) \right\} y = 0, \tag{1}$$

where $\theta = z(d/dz)$ and where the a_t and c_t are any constants, real or complex, the only restriction being that one of the c_t must be equal to unity. Such solutions can be found in a number of places in the literature.† But in attempting to study the logarithmic cases of the problem treated in the above-mentioned paper, the author was unable to find the logarithmic solutions of equation (1) in the literature. It is the purpose of this paper to present these logarithmic solutions, but for the sake of completeness, the non-logarithmic solutions are also given. The methods used are those of Frobenius.‡

2. **Non-logarithmic solutions.** The solutions of equation (1) about the point $z=0$ are all non-logarithmic in character if no two of the c_t are equal or differ by an integer; and even if some of the c_t are equal or differ by an integer, the solutions will continue to be non-logarithmic provided certain of the c_t are equal to or differ from certain of the a_t by an integer. Since these special cases can easily be recognized, we shall avoid them in our theorems by making the hypotheses stronger than necessary.

THEOREM 1. *If no two of the c_t are equal or differ by an integer, then the solutions of equation* (1) *about the point $z=0$ are non-logarithmic in character and may be written in the form*

* F. C. Smith, *Relations among the fundamental solutions of the generalized hypergeometric equation when $p=q+1$. I. Non-logarithmic cases*, this Bulletin, vol. 44 (1938), pp. 429–433.

† See, for example, L. Pochhammer, *Ueber die Differentialgleichung der allgemeineren hypergeometrischen Reihe mit zwei endlichen singulären Punkten*, Journal für die reine und angewandte Mathematik, vol. 102 (1888), pp. 76–159.

‡ G. Frobenius, *Ueber die Integration der linearen Differentialgleichungen durch Reihen*, Journal für die reine und angewandte Mathematik, vol. 76 (1873), pp. 214–235.

$$(2)\qquad Y_{0j} = z^{1-c_j}\prod_{t=1}^{q+1} \frac{\Gamma(1+c_t-c_j)}{\Gamma(1+a_t-c_j)} \sum_{n=0}^{\infty}\prod_{t=1}^{q+1} \frac{\Gamma(1+a_t-c_j+n)}{\Gamma(1+c_t-c_j+n)} z^n,$$
$$j = 1, 2, \cdots, q+1;\ |z| < 1.$$

PROOF. If we substitute into equation (1) the series

$$(3)\qquad Y_0(w) = \sum_{n=0}^{\infty} \alpha_n z^{w+n},$$

we obtain, since $f(\theta)z^n = z^n f(n)$,

$$\sum_{n=0}^{\infty} \alpha_n \left\{ \prod_{t=1}^{q+1} (w+n+a_t)z^{w+n} - \prod_{t=1}^{q+1} (w+n+c_t-1)z^{w+n-1} \right\}$$
$$(4)\qquad = \sum_{n=1}^{\infty} \left\{ \alpha_{n-1}\prod_{t=1}^{q+1} (w+n+a_t-1) - \alpha_n \prod_{t=1}^{q+1} (w+n+c_t-1) \right\} z^{w+n-1}$$
$$- \alpha_0 \prod_{t=1}^{q+1} (w+c_t-1) z^{w-1} = 0.$$

Thus, the indicial equation becomes

$$(5)\qquad \prod_{t=1}^{q+1} (w+c_t-1) = 0,$$

whose roots are

$$(6)\qquad w = 1 - c_j, \qquad j = 1, 2, \cdots, q+1.$$

Moreover, the coefficients α_n satisfy the recurrence formula

$$(7)\qquad \alpha_n = \prod_{t=1}^{q+1} \frac{(w+n+a_t-1)}{(w+n+c_t-1)} \alpha_{n-1},$$

which leads to the final result

$$(8)\qquad \begin{aligned} \alpha_n &= \prod_{t=1}^{q+1} \frac{(w+n+a_t-1)\cdots(w+a_t)}{(w+n+c_t-1)\cdots(w+c_t)} \alpha_0 \\ &= \prod_{t=1}^{q+1} \frac{\Gamma(w+c_t)\Gamma(w+a_t+n)}{\Gamma(w+a_t)\Gamma(w+c_t+n)} \alpha_0. \end{aligned}$$

If we take $\alpha_0 = 1$ and use (8) in (3), we have

$$(9)\qquad Y_0(w) = z^w \prod_{t=1}^{q+1} \frac{\Gamma(w+c_t)}{\Gamma(w+a_t)} \sum_{n=0}^{\infty} \prod_{t=1}^{q+1} \frac{\Gamma(w+a_t+n)}{\Gamma(w+c_t+n)} z^n,$$

from which the various solutions may be obtained by setting w equal to the roots (6) of equation (5). This leads to the desired result (2).

In a similar manner, we may prove the following theorem:

THEOREM 2. *If no two of the a_t are equal or differ by an integer, then the solutions of equation* (1) *about the point* $z=\infty$ *are non-logarithmic in character and may be written in the form*

$$(10)\quad Y_{\infty j} = z^{-a_j}\prod_{t=1}^{q+1}\frac{\Gamma(1-a_t+a_j)}{\Gamma(1-c_t+a_j)}\sum_{n=0}^{\infty}\prod_{t=1}^{q+1}\frac{\Gamma(1-c_t+a_j+n)}{\Gamma(1-a_t+a_j+n)}\frac{1}{z^n},$$
$$j = 1, 2, \cdots, q+1;\ |z| > 1.$$

3. **Logarithmic solutions.** If we suppose that r of the c_t are equal or differ by an integer, and assume at the same time that none of these r c_t are equal to or differ from any of the a_t by an integer, then the proof of Theorem 1 breaks down, since a zero factor will appear in the denominator of (7) for some values of n. Under these conditions, r of the solutions of (1) about the point $z=0$ become logarithmic in character. There is no loss of generality in taking the r c_t's as $c_1, c_2, \cdots, c_r$, arranged with their real parts in ascending order; thus, let us assume that

$$(11)\qquad c_2 - c_1 = l_1,\ c_3 - c_2 = l_2,\ \cdots,\ c_r - c_{r-1} = l_{r-1},$$

where each l_v is zero or a positive integer. Under these conditions we may state the following theorem:

THEOREM 3. *If $c_1, c_2, \cdots, c_r$ satisfy* (11) *but do not equal or differ from any of the a_t by an integer, then the solutions Y_{0j}, $(j=1, r+1, \cdots, q+1)$, of equation* (1) *are given by* (2), *but the remaining Y_{0j} are logarithmic in character and may be written in the form*

$$(12)\quad \begin{aligned} Y_{0j} &= (-1)^{j+1}\sum_{v=1}^{j-1}(-1)^v C_{j-1,v-1}(\log z)^{j-v}Y_{0v} \\ &\quad + \sum_{v=1}^{j} z^{1-c_v}\frac{(j-1)!}{(j-v)!}G_v^{(j-v)}(0, z),\quad j = 2, \cdots, r;^{*}\ |z| < 1, \end{aligned}$$

where $G_v^{(j-v)}(0, z)$ denotes the $(j-v)$th derivative with respect to w of the function

* If we agree to delete the first summation of (12) when $j=1$, then Y_{01} can also be obtained from (12).

$$G_v(w, z) = (-1)^{1-v-\sum_{t=1}^{v-1} tl_t}\left(\frac{\pi w}{\sin \pi w}\right)^{1-v} \prod_{t=1}^{q+1} \frac{\Gamma(1+c_t-c_1+w)}{\Gamma(1+a_t-c_1+w)}$$

$$\cdot \sum_{n=0}^{l_{v-1}-1} \prod_{t=1}^{v-1} \Gamma(c_v - c_t - w - n)\Gamma(1+a_t-c_v+w+n) \tag{13}$$

$$\cdot \prod_{t=v}^{q+1} \frac{\Gamma(1+a_t-c_v+w+n)}{\Gamma(1+c_t-c_v+w+n)} [(-1)^{v-1}z]^n,$$

evaluated for $w=0$; *in* (13) *the* -1 *factor and the first product of the summation are to be deleted when* $v=1$ *and the special definition* $l_0=\infty$ *is to be taken*; *moreover, if* $l_{v-1}=0$, *the special convention* $G_v(w, z)=0$ *is made.*

PROOF. According to the theory of Frobenius, the solutions Y_{0j}, $(j=1, \cdots, r)$, of equation (1) may be obtained by setting $w=1-c_r$ in

$$V(w) = K(w)(w+c_r-1)^{r-1}Y_0(w), V'(w), V''(w), \cdots; V^{(r-1)}(w), \tag{14}$$

in which $Y_0(w)$ is given by (9), and in which $K(w)$ is an arbitrary analytic function of w for $w=1-c_r$. Now $V(w)$ may be written

$$\begin{aligned} V(w) &= \frac{K(w)z^w \prod_{t=r}^{q+1} \Gamma(w+c_t)}{\prod_{t=1}^{q+1} \Gamma(w+a_t)} \left\{ (w+c_r-1)^{r-1} \prod_{t=1}^{r-1} \Gamma(w+c_t) \right. \\ &\quad \left. \cdot \sum_{n=0}^{\infty} \prod_{t=1}^{q+1} \frac{\Gamma(w+a_t+n)}{\Gamma(w+c_t+n)} z^n \right\} \\ &= \frac{K(w)z^{w+c_r} \prod_{t=r}^{q+1} \Gamma(w+c_t)}{\prod_{t=1}^{q+1} \Gamma(w+a_t)} \left\{ (w+c_r-1)^{r-1} \prod_{t=1}^{r-1} \Gamma(w+c_t) \right. \\ &\quad \left. \cdot \sum_{v=1}^{r} z^{-c_v} \sum_{n=0}^{l_{v-1}-1} \prod_{t=1}^{q+1} \frac{\Gamma(w+a_t+c_r-c_v+n)}{\Gamma(w+c_t+c_r-c_v+n)} z^n \right\}. \end{aligned} \tag{15}$$

By several applications of the well known relation

$$\Gamma(w) = \frac{\pi}{\Gamma(1-w)\sin \pi w}, \tag{16}$$

we may reduce (15) to the form

$$V(w) = \frac{K(w) z^{w+c_r} \prod_{t=r}^{q+1} \Gamma(w + c_t)}{\prod_{t=1}^{q+1} \Gamma(w + a_t) \prod_{t=1}^{r-1} \Gamma(1 - c_t - w)}$$

$$(17) \qquad \cdot \left\{ \sum_{v=1}^{r} z^{-c_v} (-1)^{r-1+\sum_{t=1}^{r-1} t l_t} (w + c_r - 1)^{v-1} \right.$$

$$\cdot \left[\frac{\pi(w + c_r - 1)}{\sin \pi(w + c_r - 1)} \right]^{r-v}$$

$$\left. \cdot \prod_{t=1}^{q+1} \frac{\Gamma(w + c_r + c_t - c_1)}{\Gamma(w + c_r + a_t - c_1)} G_v(w + c_r - 1, z) \right\}.$$

Since $K(w)$ is an arbitrary analytic function of w for $w = 1 - c_r$, we may choose it so that (17) reduces to the form

$$(18) \quad V(w) = z^{w+c_r} \left\{ \sum_{v=1}^{r} z^{-c_v} (w + c_r - 1)^{v-1} G_v(w + c_r - 1, z) \right\}.$$

Thus,

$$(19) \qquad V(w - c_r + 1) = z^w \left\{ \sum_{v=1}^{r} z^{1-c_v} w^{v-1} G_v(w, z) \right\}.$$

According to the theory of Frobenius, then, we have

$$Y_{0j} = \left[\frac{\partial^{j-1} V(w)}{\partial w^{j-1}} \right]_{w=1-c_r} = \left[\frac{\partial^{j-1} V(w - c_r + 1)}{\partial w^{j-1}} \right]_{w=0}$$

$$= \sum_{v=1}^{r} z^{1-c_v} \left[\frac{\partial^{j-1}}{\partial w^{j-1}} z^w w^{v-1} G_v(w, z) \right]_{w=0}$$

$$(20) \qquad = \sum_{v=1}^{r} z^{1-c_v} \sum_{h=0}^{j-1} C_{j-1,h} \left[\frac{\partial^{j-h-1}}{\partial w^{j-h-1}} z^w G_v(w, z) \right]_{w=0}$$

$$\cdot \left[(v-1) \cdots (v-h) w^{v-h-1} \right]_{w=0}$$

$$= \sum_{v=1}^{j} z^{1-c_v} \frac{(j-1)!}{(j-v)!} \left[\frac{\partial^{j-v}}{\partial w^{j-v}} z^w G_v(w, z) \right]_{w=0}, \qquad j = 1, 2, \cdots, r.$$

In order to obtain the desired result (12) from (20), we give a proof by induction. First of all, we note that

$$(21) \qquad \begin{aligned} Y_{02} &= z^{1-c_1} \left[\frac{\partial}{\partial w} z^w G_1(w, z) \right]_{w=0} + z^{1-c_2} G_2(0, z) \\ &= \log z \, Y_{01} + z^{1-c_1} G_1'(0, z) + z^{1-c_2} G_2(0, z), \end{aligned}$$

which is the desired result (12) for $j=2$. To complete the proof, we assume that (12) holds for $Y_{02}, \cdots, Y_{0,j-1}$ and then show that it also holds for Y_{0j}. By the theorem of Leibnitz, we have from (20)

$$
\begin{aligned}
Y_{0j} &= \sum_{v=1}^{j} z^{1-c_v} \frac{(j-1)!}{(j-v)!} \sum_{k=0}^{j-v} C_{j-v,k}(\log z)^k G_v^{(j-v-k)}(0, z) \\
&= \sum_{k=1}^{j} (\log z)^{j-k} \sum_{v=1}^{k} z^{1-c_v} \frac{(j-1)!}{(j-v)!} C_{j-v,j-k} G_v^{(k-v)}(0, z) \\
&= \sum_{k=1}^{j-1} C_{j-1,k-1}(\log z)^{j-k} \sum_{v=1}^{k} z^{1-c_v} \frac{(k-1)!}{(k-v)!} G_v^{(k-v)}(0, z) \\
&\quad + \sum_{v=1}^{j} z^{1-c_v} \frac{(j-1)!}{(j-v)!} G_v^{(j-v)}(0, z).
\end{aligned} \tag{22}
$$

The second summation here agrees with the second summation of (12). In order to show that the two first summations agree, we make use of our above assumption that

$$
\begin{aligned}
Y_{0k} &= (-1)^{k+1} \sum_{v=1}^{k-1} (-1)^v C_{k-1,v-1}(\log z)^{k-v} Y_{0v} \\
&\quad + \sum_{v=1}^{k} z^{1-c_v} \frac{(k-1)!}{(k-v)!} G_v^{(k-v)}(0, z), \quad 2 \leqq k \leqq j-1,
\end{aligned} \tag{23}
$$

so that

$$
\begin{aligned}
&\sum_{v=1}^{k} z^{1-c_v} \frac{(k-1)!}{(k-v)!} G_v^{(k-v)}(0, z) \\
&\quad = Y_{0k} + (-1)^k \sum_{v=1}^{k-1} (-1)^v C_{k-1,v-1}(\log z)^{k-v} Y_{0v} \\
&\quad = (-1)^k \sum_{v=1}^{k} (-1)^v C_{k-1,v-1}(\log z)^{k-v} Y_{0v}.
\end{aligned} \tag{24}
$$

When (24) is substituted into the first summation of (22), we obtain

$$
\begin{aligned}
&\sum_{k=1}^{j-1} C_{j-1,k-1}(\log z)^{j-k}(-1)^k \sum_{v=1}^{k} (-1)^v C_{k-1,v-1}(\log z)^{k-v} Y_{0v} \\
&\quad = \sum_{k=1}^{j-1} (-1)^k C_{j-1,k-1} \sum_{v=1}^{k} (-1)^v C_{k-1,v-1}(\log z)^{j-v} Y_{0v} \\
&\quad = \sum_{v=1}^{j-1} (-1)^v (\log z)^{j-v} Y_{0v} \sum_{k=v}^{j-1} (-1)^k C_{j-1,k-1} C_{k-1,v-1}
\end{aligned} \tag{25}
$$

$$= \sum_{v=1}^{j-1} C_{j-1,v-1}(\log z)^{j-v} Y_{0v} \sum_{k=0}^{j-v-1} (-1)^k C_{j-v,k}.$$

If, in the binomial expansion of $(a+b)^{j-v}$, we set $a=1$ and $b=-1$, we obtain

$$0 = \sum_{k=0}^{j-v} (-1)^k C_{j-v,k} = (-1)^{j-v} + \sum_{k=0}^{j-v-1} (-1)^k C_{j-v,k}, \tag{26}$$

from which

$$\sum_{k=0}^{j-v-1} (-1)^k C_{j-v,k} = (-1)^{j-v+1} = (-1)^{j+v+1}. \tag{27}$$

When (27) is used in the last member of (25), we obtain the desired first summation of (12). This completes the proof of Theorem 3.

If we assume that

$$a_1 - a_2 = k_1,\ a_2 - a_3 = k_2,\ \cdots,\ a_{s-1} - a_s = k_{s-1}, \tag{28}$$

where each k_v is zero or a positive integer, then, by means of a proof similar to that given above, we may establish the following theorem:

THEOREM 4. *If $a_1, a_2, \cdots, a_s$ satisfy* (28) *but do not equal or differ from any of the c_t by an integer, then the solutions $Y_{\infty j}$, $(j=1, s+1, \cdots, q+1)$, of equation* (1) *are given by* (10), *but the remaining $Y_{\infty j}$ are logarithmic in character and may be written in the form*

$$\begin{aligned} Y_{\infty j} &= (-1)^{j+1} \sum_{v=1}^{j-1} (-1)^v C_{j-1,v-1} \left(\log \frac{1}{z}\right)^{j-v} Y_{\infty v} \\ &\quad + \sum_{v=1}^{j} z^{-a_v} \frac{(j-1)!}{(j-v)!} F_v^{(j-v)}(0, z), \quad j = 2, 3, \cdots, s;^* \ |z| > 1, \end{aligned} \tag{29}$$

where $F_v^{(j-v)}(0, z)$ denotes the $(j-v)$th derivative with respect to w of the function

$$\begin{aligned} F_v(w, z) &= (-1)^{1-v-\sum_{t=1}^{v-1} t k_t} \left(\frac{\pi w}{\sin \pi w}\right)^{1-v} \prod_{t=1}^{q+1} \frac{\Gamma(1 - a_t + a_1 + w)}{\Gamma(1 - c_t + a_1 + w)} \\ &\quad \cdot \sum_{n=0}^{k_{v-1}-1} \prod_{t=1}^{v-1} \Gamma(a_t - a_v - w - n)\Gamma(1 - c_t + a_v + w + n) \\ &\quad \cdot \prod_{t=v}^{q+1} \frac{\Gamma(1 - c_t + a_v + w + n)}{\Gamma(1 - a_t + a_v + w + n)} \left[(-1)^{v-1} \frac{1}{z}\right]^n, \end{aligned} \tag{30}$$

* Again, if we agree to delete the first summation of (29) when $j=1$, then $Y_{\infty 1}$ can also be obtained from (29).

evaluated for $w=0$; *in* (30) *we make special conventions of the same type as those made in connection with* (13).

In connection with Theorem 4, it is of interest to note the unexpanded forms corresponding to (20), namely,

$$Y_{\infty j} = \sum_{v=1}^{j} z^{-a_v} \frac{(j-1)!}{(j-v)!} \left[\frac{\partial^{j-v}}{\partial w^{j-v}} z^{-w} F_v(w, z) \right]_{w=0}, \tag{31}$$

$$j = 1, 2, \cdots, s.$$

The College of St. Francis

ON THE FIRST CASE OF FERMAT'S LAST THEOREM*

BARKLEY ROSSER

We prove the following theorem:

Theorem. *If p is an odd prime, α, β, and γ are integers in the field of the pth roots of unity, $\alpha\beta\gamma$ is prime to p, and*

$$\alpha^p + \beta^p + \gamma^p = 0,$$

then $p \geqq 8,332,403$.

As ordinary integers are integers in the field of the pth roots of unity, we infer the following:

Corollary. *The equation*

$$x^p + y^p + z^p = 0$$

has no solution in integers prime to p if p is an odd prime less than 8,332,403.

To abbreviate statements, we shall say that an odd prime p is improper if there are integers α, β, and γ in the field of the pth roots of unity such that $\alpha\beta\gamma$ is prime to p and

$$\alpha^p + \beta^p + \gamma^p = 0.$$

Then the theorem to be proved can be stated in the form:

Theorem. *There are no improper odd primes less than* 8,332,403.

The proof is based on a theorem of Morishima† which, in our

* Presented to the Society, February 25, 1939.

† Taro Morishima, *Über die Fermatsche Vermutung*, Japanese Journal of Mathematics, vol. 11 (1935), pp. 241–252. Earlier results of a similar nature are due to Pollaczek, Frobenius, Vandiver, Mirimanoff, and Wieferich. Compare Dickson's *History of the Theory of Numbers.*