CARTAN ON GROUPS AND DIFFERENTIAL GEOMETRY

La Théorie des Groupes Finis et Continus et la Géométrie Différentielle traitées par la Méthode du Repère Mobile. By Élie Cartan. (Cahiers Scientifiques, no. 18.) Paris, Gauthier-Villars, 1937. 6+269 pp.

This book, which originated from a course of lectures given in 1931–1932 at the Sorbonne, covers in a somewhat more explicit form essentially the same material as no. 194 (1935) of the Actualités Scientifiques et Industrielles (see the review, this Bulletin, vol. 41 (1935), p. 774). By means of the method of the repère mobile the author studies arbitrary manifolds M_λ in a Klein space R whose geometry is described by its group of automorphisms. The chief aim of this review shall be to bring out the axiomatic foundations of the theory.

Coördinatization of a space R consists in a one-to-one mapping of the points A of R upon a manifold Σ of (numerical) symbols x serving as coördinates. In a Klein space such coördinatization is possible only with respect to a frame of reference, or briefly frame, $\mathfrak{f}$. An abstract group G and a realization of it by means of one-to-one transformations of Σ are supposed to be given. We thus deal with four kinds of objects: points A, symbols x, frames $\mathfrak{f}$, and group elements s, their mutual relation being established by two axioms (a) and (b):

(a) Any pair of frames $\mathfrak{f}, \mathfrak{f}'$ determines a group element $s=(\mathfrak{f}\rightarrow\mathfrak{f}')$ called the transition from $\mathfrak{f}$ to $\mathfrak{f}'$. Vice versa, any element s of G carries a given frame $\mathfrak{f}$ into a uniquely determined frame $\mathfrak{f}'$ such that $s=(\mathfrak{f}\rightarrow\mathfrak{f}')$. Succession of transitions $\mathfrak{f}\rightarrow\mathfrak{f}'\rightarrow\mathfrak{f}''$ corresponds to the composition of group elements:

$$s = (\mathfrak{f} \rightarrow \mathfrak{f}'), \quad t = (\mathfrak{f}' \rightarrow \mathfrak{f}'') \quad \text{imply} \quad ts = (\mathfrak{f} \rightarrow \mathfrak{f}'').$$

(The identical element is $\mathfrak{f}\rightarrow\mathfrak{f}$, and $\mathfrak{f}'\rightarrow\mathfrak{f}$ is the inverse of $\mathfrak{f}\rightarrow\mathfrak{f}'$.)

(b) With respect to a given frame $\mathfrak{f}$ each point A determines a symbol $x=(A, \mathfrak{f})$ as its coördinate, thus setting up a one-to-one correspondence $A\rightleftarrows x$ between R and Σ. The coördinate $x'=(A, \mathfrak{f}')$ of A in another frame $\mathfrak{f}'$ arises from x by the transformation associated with the group element $s=(\mathfrak{f}\rightarrow\mathfrak{f}')$ in the given realization.

Consequences: $\mathfrak{f}$, $\mathfrak{f}^*$ being any two frames, the equation $(A^*, \mathfrak{f}^*)=(A, \mathfrak{f})$ defines a one-to-one mapping $A\rightarrow A^*$ of R upon itself, the *space automorphism* $\{\mathfrak{f}, \mathfrak{f}^*\}$. If a group element t changes $\mathfrak{f}$, $\mathfrak{f}^*$ into $\mathfrak{g}, \mathfrak{g}^*$, one evidently has at the same time $(A^*, \mathfrak{g}^*) =(A, \mathfrak{g})$. Hence the space automorphisms $\{\mathfrak{f}, \mathfrak{f}^*\}$ form a group isomorphic with G; but their isomorphic mapping onto the elements of G is fixed except for an arbitrary inner automorphism of the group G. Figures in R which arise from each other by space automorphisms are considered *equal.*

We are concerned with λ-dimensional parametrized manifolds

$$M_\lambda: \quad x = x(t_1, \cdots, t_\lambda),$$

where t_α are real parameters. Let us employ for a moment ordinary real coördinates in R, $x=(x_1, \cdots, x_n)$. At a given point $A=(t_1, \cdots, t_\lambda)$ of M_λ the functions $x_i(t_1, \cdots, t_\lambda)$ and their derivatives up to a given order p constitute a contact element of order p, or briefly a p-spread. We obtain a succession of such spreads of orders $p=0, 1, 2, \cdots$, each of which is contained in the following. The central problem consists in deciding when two parametrized manifolds M_λ, M_λ' are equal. In the analytic case one may ask instead under what circumstances two given p-spreads are equal, and then apply such a criterion to the aforementioned succession of spreads of orders $p=0, 1, 2, \cdots$. We start with the lowest case, $p=0$, equality of two points.

All symbols arising from a given x by the operations of our group form a *layer* $\phi=\Sigma(x)$ in Σ. The *point* A belongs to ϕ if its coördinate with respect to some, and hence to every, frame $\mathfrak{f}$ lies in ϕ; in a manner independent of the frames the stratification of Σ transfers to R. Let us assume that Σ, when considered as the manifold of its layers, is m-dimensional and that $k_1, \cdots, k_m$ are parameters whose simultaneous values characterize and distinguish the individual layers within this manifold. They can be used as point invariants (invariants of order 0) thus solving the central problem for points. Cartan demands construction in the simplest possible way of a manifold Σ_0 in Σ intersecting each layer at exactly one point x_0; $k_1, \cdots, k_m$ are introduced as parameters on Σ_0.

With every x_0 on Σ_0 there is associated the subgroup $G^0=G(x_0)=G(k_1, \cdots, k_m)$ of G whose elements are realized by transformations leaving x_0 fixed. We assume G to be an r-parameter continuous Lie group, and we denote by $\omega_1, \cdots, \omega_r$ a basis for the components of the generic infinitesimal element of G. Moreover, we ascertain a basis

$$\pi_i = c_{i1}(k)\omega_1 + \cdots + c_{ir}(k)\omega_r, \qquad i = 1, \cdots, m,$$

for those components whose vanishing characterizes the infinitesimal elements of $G(x_0)$; Cartan calls them the *principal components of order* 0. For a point A belonging to the layer $\Sigma(x_0)$ the *frames* $\mathfrak{f}$ *of order* 0 are introduced by the requirement

$$(A, \mathfrak{f}) = x_0.$$

Transition among frames of order 0 is accomplished by means of the elements of the subgroup $G(x_0)$.

From a single point A we pass on to a pair of points AA', or more specifically to a line element AA' issuing from A (A' infinitely near A), asking again when two such line elements AA', BB' are equal. Let A, A' belong to the layers $\Sigma(x_0)$, $\Sigma(x_0')$, respectively; the frames $\mathfrak{f}$, $\mathfrak{f}'$ shall be restricted by the equations

$$(A, \mathfrak{f}) = x_0, \qquad (A', \mathfrak{f}') = x_0'. \tag{1}$$

We want to know when there exists a frame $\mathfrak{g}$ such that

$$(A, \mathfrak{f}) = (B, \mathfrak{g}), \qquad (A', \mathfrak{f}) = (B', \mathfrak{g}).$$

We introduce the frame $\mathfrak{g}'$ arising from $\mathfrak{g}$ by the transition $\omega=(\mathfrak{f}\to\mathfrak{f}')$ and then obtain

$$(A, \mathfrak{f}) = x_0, \qquad (A', \mathfrak{f}') = x_0' \;\Big|\; (B, \mathfrak{g}) = x_0, \qquad (B', \mathfrak{g}') = x_0',$$
$$(\mathfrak{f} \to \mathfrak{f}') = (\mathfrak{g} \to \mathfrak{g}').$$

A full system of invariantive characteristics of AA' therefore consists of the values k_i, $k_i' = k_i + dk_i$ of the invariants at the points A and A', and of the set of transitions $\omega = (\mathfrak{f}\to\mathfrak{f}')$ among frames $\mathfrak{f}$, $\mathfrak{f}'$ satisfying (1). We choose $\mathfrak{f}'$ infinitely near to $\mathfrak{f}$ so that ω is infinitesimal. With $\mathfrak{f}$ fixed, the variation of $\mathfrak{f}'$ leaves the m components π_i untouched while assigning perfectly arbitrary values to the remaining $r-m$ components. If one passes from $\mathfrak{f}$, $\mathfrak{f}'$ to another pair by means of group elements s, $s+ds$ in $G(x_0)$, $G(x_0')$, respectively, then ω changes into

$$\bar{\omega} = s^{-1}\omega s + s^{-1}ds.$$

In other words, the π_i undergo a nonhomogeneous linear substitution

$$\bar{\pi}_i = \sum_k \sigma_{ik}\pi_k + \delta\sigma_i$$

depending on s only.

We apply our remark to all points $A'=(t_\alpha+dt_\alpha)$ in the neighborhood of a given point $A=(t_\alpha)$ on a parametrized manifold M_λ. On emphasizing dependence on the line element (dt_α) we shall get equations

$$dk_i = k_{i1}dt_1 + \cdots + k_{i\lambda}dt_\lambda, \qquad i = 1, \cdots, m, \tag{2}$$

and with respect to an arbitrary frame $\mathfrak{f}$ of order 0 in A,

$$\pi_i = b_{i1}dt_1 + \cdots + b_{i\lambda}dt_\lambda. \tag{3}$$

The matrix $\|b_{i\alpha}\|$ appears here as coördinate in a certain Klein space $R^{(1)}$ with the group $G(x_0)=G^0$, each element s of G^0 being realized by a transformation of the coördinate of the special type

$$\bar{b}_{i\alpha} = \sum_k \sigma_{ik}b_{k\alpha} + \overset{0}{\sigma}_{i\alpha}.$$

With the new Klein space we may proceed as before, and we then will obtain a number of invariants $k_j^{(1)}$ of order 1 for our 1-spreads, a subgroup $G^{(1)}$ of G^0 depending on the values of the new invariants, and the "frames of order 1" forming a subclass of the frames of order 0 and turning into each other by means of the elements of $G^{(1)}$. A basis for those components of the generic infinitesimal element of G whose vanishing characterizes the elements of $G^{(1)}$ may be ascertained by adding to $\pi_1, \cdots, \pi_m$ some further components

$$\overset{(1)}{\pi_j} = \overset{(1)}{c_{j1}}\omega_1 + \cdots + \overset{(1)}{c_{jr}}\omega_r. \tag{4}$$

The full invariantive characterization of a 1-spread consists in giving the values of k_i, $k_{i\alpha}$, and $k_j^{(1)}$. The coefficients $c^{(1)}$ in (4) depend on these arguments.

In the same fashion as from 0 to 1, one passes from order $p-1$ to p. The shrinkage of the group $G \supset G^0 \supset G^{(1)} \supset \cdots$ must come to a standstill after a finite number of steps, and with the next step the production of new invariants will stop. On a parametrized manifold M_λ all these invariants are functions of the parameters t_α. Their coincidence for two such manifolds M_λ, M_λ' is necessary and sufficient for the manifolds to be equal. This unicity theorem is accompanied by a corresponding existence theorem. For one-dimensional manifolds the invariants are subject to no restriction, while in the case of several parameters t_α certain integrability conditions must be satisfied.

It is possible, and becomes sometimes desirable, to investigate special classes of manifolds by imposing conditions on the invariants.

Curves in euclidean 3-space E_3 are one of the simplest examples of our theory. With respect to a cartesian frame, we have triples of numbers (x_1, x_2, x_3) as coördinates; the space is homogeneous, and Σ_0 consists of the one symbol $(0, 0, 0)$. The frames with the point A of the curve as their origin are of order 0; those whose first axis, moreover, coincides with the tangent, are of order 1; and finally the Frenet trihedral is the one frame of order 2. Here the process comes to a standstill. There is no invariant of order 0, but there is one each of the orders 1, 2, 3, namely ds/dt (t being the parameter, ds the arc element), curvature ρ, and torsion τ, respectively. For the minimal curves in the (complex) E_3 for which the invariant ds/dt vanishes, the above normalization of the frames of order 1 becomes impossible; they therefore require a separate treatment illustrating our remark in the preceding paragraph. The same is true for the plane curves with $\rho=0$.

Slightly more complicated is the study of manifolds without a fixed parametrization. One then wants to know when two manifolds,

$$M_\lambda:\ x = \phi(t_1, \cdots, t_\lambda) \qquad \text{and} \qquad M_\lambda:\ x = \phi'(\tau_1, \cdots, \tau_\lambda),$$

may be changed into each other by a suitable automorphism combined with a suitable transformation of the parameters. Both influences, of the arbitrary frame $\mathfrak{f}$ and of the arbitrary parameters t_α, have to be taken into account. In that case one must combine the coefficients $k_{i\alpha}$ with the $b_{i\alpha}$ in (2) and (3) as coördinates in a Klein space $R^{(1)}$, since the transformation of the t_α affects the $b_{i\alpha}$ as well as the $k_{i\alpha}$ (although the latter are indifferent towards a change of frame). It is this problem with which Cartan deals in the present book, and in some way he reduces the second influence, the choice of parameters, to the choice of the frame. I did not quite understand how he does this in general, though in the examples he gives the procedure is clear. To me it seems advisable to keep both factors apart from the beginning; the process itself tends to normalize both in mutual interdependence as it advances to higher and higher orders. The same situation is met with everywhere in differential geometry. For instance, riemannian spaces could be treated by introducing coördinates and attaching to each point A a frame, that is, a cartesian set of axes. Invariance is required with respect to arbitrary transformations of the coördinates and to orthogonal transformations of the frames which may depend arbitrarily on the point A. One knows how Gauss, Riemann, and Einstein got around the frames: the parameters once chosen define uniquely at each point an affine set of axes, and one takes advantage of it by treating cartesian geometry in terms of affine frames and a fundamental metric form rather than in terms of cartesian frames. Cartan goes here to the opposite extreme by normalizing the parameters in terms of the frames. I should advocate full impartiality on both sides as long as one deals with fairly general differential geometric problems.

The book under review pursues a three-fold purpose: it contains (1) an exposition of the general theory of finite continuous Lie groups in a terminology adapted to its differential geometric applications; (2) a general description of the method of repères mobiles; and (3) its application to a number of important examples. The arrangement is didactic rather than systematic. Thus the first examples, Chapters 1–3, on curves in E_3, minimal curves in E_3, ruled surfaces in E_3 (considered as one-dimensional manifolds of lines), precede the general formulation. Chapters 4–9, 11, 13, 14, deal with Lie groups. While the topics (1) and (3) are given in full detail, the central problem (2) emphasized by this review is but briefly dealt with, at the beginning of Chapter 10 more from the standpoint of transformation groups, at the beginning of Chapter 12 more from the abstract standpoint. In both chapters there follow further applications: curves in the affine and the projective planes, and arbitrary surfaces in E_3. With the last example, the only one concerned with manifolds of more than one dimension, the integrability conditions turn up; although their rôle in the theory of Lie groups is amply discussed, their general formulation as an intrinsic part of the existence theorem in the theory of repèrage is omitted.

All of the author's books, the present one not excepted, are highly stimulating, full of original viewpoints, and profuse in interesting geometric details. Cartan is undoubtedly the greatest living master of differential geometry. This review is incomplete in so far as it has tried to lay bare the roots, rather than describe the rich foliage of the tree which his book unfolds before its reader. We should not let pass unmentioned Jean Leray's merit in molding the lecture notes he took into something which is a true book and yet catches some of the vividness of the original lectures. Nevertheless, I must admit that I found the book, like most of Cartan's papers, hard reading. Does the reason lie only in the great French geometric tradition on which Cartan draws, and the style and contents of which he takes more or less for granted as a common ground for all geometers, while we, born and educated in other countries, do not share it?

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