ON THE EXTENDED FORM OF CAUCHY'S CONDENSATION TEST FOR THE CONVERGENCE OF INFINITE SERIES

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Cauchy's celebrated theorem,* which forms the basis of some of the most powerful chain tests for the convergence and divergence of infinite series, notably the logarithmic scale and DeMorgan and Bertrand's T and τ scales, is as follows:

Let $f(n)$ represent any positive continuous and single-valued function of n which is monotone decreasing as n increases. Suppose, moreover, that $\lim_{n\to\infty} f(n)=0$. Then $\sum^\infty f(n)$ and $\sum^\infty a^n f(a^n)$ are both convergent or both divergent, provided a is a positive integer not less than 2.

The restriction $a \not< 2$ was removed by Kohn† in 1882. The proof covers several pages. Kohn showed that the theorem holds for any real value of a greater than unity.

A shorter proof of Kohn's extension of Cauchy's theorem was given by Hill‡ in 1896. Hill's proof depends on two auxiliary theorems which, in turn, are based on the two previously established inequalities

$$\left(1+\frac{1}{a}+\frac{1}{a^{s+1}}\right)t_{s+1} < t_{s+1} - t_s,$$

$$t_{s+1} - t_s < \left(a-1+\frac{1}{a^s}\right)(1+t_s),$$

where t_s is the integral part of a^s.

In view of the fundamental importance of Cauchy's theorem the following simple analytical proof of the theorem with Kohn's extension may, therefore, be of some interest. The proof which follows is in fact in the nature of a corollary to the well known integral test which may be stated as follows:

Under the assumptions set forth in Cauchy's theorem, $\sum^\infty f(n)$ and $\int^\infty f(x)dx$ converge and diverge together.

Set $x=a^u$, $(a>1)$; then $\log a>0$ and

* Cauchy, Anciens Exercises, vol. 2 (1827), p. 221.

† Kohn, G., Grunert Archiv, vol. 67 (1882), pp. 63–95.

‡ Hill, M. J. M., Messenger of Mathematics, (2), vol. 26 (1896–1897), p. 102.

$$\int^{\infty} f(x)dx = \log a \int^{\infty} a^u f(a^u)du = \log a \int^{\infty} a^x f(a^x)dx.$$

Therefore $\sum^{\infty} f(n)$ converges and diverges with $\int^{\infty} a^x f(a^x)dx$, and it remains only to show that $\int^{\infty} a^x f(a^x)dx$ converges and diverges with $\sum^{\infty} a^n f(a^n)$.

For values of x between the two integers n and $n+1$, we have, if a is greater than unity, $a^n < a^x < a^{n+1}$; and since $f(x)$ is monotone decreasing,

$$f(a^{n+1}) < f(a^x) < f(a^n).$$

Furthermore, since $f(x)$ is positive we have, on multiplying these inequalities term by term,

$$a^n f(a^{n+1}) < a^x f(a^x) < a^{n+1} f(a^n),$$

from which, on integrating with respect to x between the limits n and $n+1$, we have

$$\frac{1}{a} \cdot a^{n+1} f(a^{n+1}) < \int_n^{n+1} a^x f(a^x)dx < a \cdot a^n f(a^n)$$

and therefore

$$\frac{1}{a} \sum_{n+1}^{\infty} a^k f(a^k) < \int_n^{n+1} a^x f(a^x)dx + \int_{n+1}^{n+2} a^x f(a^x)dx + \int_{n+2}^{n+3} a^x f(a^x)dx + \cdots = \int_n^{\infty} a^x f(a^x)dx < a \sum_n^{\infty} a^k f(a^k).$$

These two inequalities make it obvious that $\sum^{\infty} a^n f(a^n)$ and $\int^{\infty} a^x f(a^x)dx$ converge and diverge together.

This completes the proof of both Cauchy's theorem and Kohn's extension.

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