closed interval with norm the absolute value of the function, and the space of all functions which are Lebesgue integrable to the pth power, $p \geqq 1$, with norm the pth root of the integral of the pth power of the absolute value of the function, are all spaces with a denumerable base in the sense of Schauder and Banach, and consequently of type A, the above theorem holds of all completely continuous linear transformations with Banach spaces as domains and such spaces as ranges.*

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MULTIVALENT FUNCTIONS OF ORDER p†

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1. **Introduction.** For the class of k-wise symmetric functions

$$(1.1)\qquad f(z) = \sum_{n=1}^{\infty} a_n z^n, \qquad a_1 = 1, \; a_n = 0 \text{ for } n \not\equiv 1 \pmod{k},$$

which are regular and univalent within the unit circle, it has been conjectured that there exists a constant $A(k)$ so that for all n

$$(1.2)\qquad |a_n| \leqq A(k) n^{2/k-1}.$$

Proofs of this inequality for $k=1, 2, 2, 3$, were given by J. E. Littlewood,§ R. E. A. C. Paley and J. E. Littlewood,|| E. Landau,¶ and V. Levin** respectively. As far as the author is aware there is no valid proof†† for $k>3$ in the literature as yet.

It is the purpose of this note to point out that the methods of proof

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* Hildebrandt, this Bulletin, vol. 36 (1931), p. 197.

† Presented to the Society, February 20, 1937.

‡ The author is indebted to the referee for helpful suggestions which led to a revision of this note.

§ See J. E. Littlewood, *On inequalities in the theory of functions*, Proceedings of the London Mathematical Society, (2), vol. 23 (1925), pp. 481–519.

|| See R. E. A. C. Paley and J. E. Littlewood, *A proof that an odd schlicht function has bounded coefficients*, Journal of the London Mathematical Society, vol. 7 (1932), pp. 167–169.

¶ See E. Landau, *Über ungerade schlichte Funktionen*, Mathematische Zeitschrift, vol. 37 (1933), pp. 33–35.

** See V. Levin, *Ein Beitrag zum Koeffizientproblem der schlichten Funktionen*, Mathematische Zeitscrift, vol. 38 (1934), pp. 306–311.

†† See K. Joh and S. Takahashi, *Ein Beweis für Szegösche Vermutung über schlichte Potenzreihen*, Proceedings of the Imperial Academy of Japan, vol. 10 (1934) pp. 137–139. The proof therein was found to be defective: see Zentralblatt für Mathematik, vol. 9 (1934), pp. 75–76.

used in obtaining the inequality (1.2) for $k=2$ can be utilized to obtain a more general inequality for functions multivalent of order p with respect to the unit circle, provided these functions in question have no zeros within the unit circle other than at the origin. More specifically, let m be a non-negative integer, and let

$$(1.3)\qquad f(z) = \sum_{1+mk}^{\infty} a_n z^n, \qquad a_{1+mk} = 1,\ a_n = 0 \text{ for } n \not\equiv 1 \pmod{k},$$

be a k-wise symmetric function, regular and p-valent within the unit circle with $f(z)\neq 0$ for $0<|z|<1$. Then for all n

$$(1.4)\qquad | a_n | < A(p,\ k) n^{2p/k-1}, \qquad k < 4p,$$

where $A(p, k)$ is a constant independent of n and $f(z)$. We note in passing that a sufficient condition that $f(z)\neq 0$ for $0<|z|<1$ is that $p<k(m+1)+1$. For, if $f(z)$ vanishes in $0<|z|<1$ it vanishes $(1+mk)+k$ times within the unit circle for the reason that $f(z)$ is k-wise symmetric. On the other hand, $f(z)$ cannot vanish more than p times within the unit circle as $f(z)$ is p-valent. The condition $f(z)\neq 0$ for $0<|z|<1$ is automatically fulfilled for the following special case which is a generalization of the Paley-Littlewood inequality ($p=1$). Let the $2p$-wise symmetric function

$$(1.5)\qquad f(z) = \sum_{n=1}^{\infty} a_n z^n, \qquad a_1 = 1,\ a_n = 0 \text{ for } n \not\equiv 1 \pmod{2p}$$

be regular and multivalent of order p within the unit circle; then the coefficients are uniformly bounded.

The inequality (1.4) for the particular case $k=1$, but general p, was shown to be true by M. Biernacki, who found that if the condition $f(z)\neq 0$ for $0<|z|<1$ be discarded, the inequality for the coefficients takes the form

$$(1.6)\qquad |a_n| < A(p)\cdot \mu_q n^{2p-1},$$

where $\mu_q = \text{maximum } \{|a_1|, |a_2|, \cdots, |a_q|\}$, q being the number of zeros of $f(z)$ located within the unit circle.*

2. **The proof for** (1.4). The following lemma for $p=1$ has been used by many authors, and is no doubt well known to many for $p>1$, though I know of no place in the literature where it has been proved.

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* See M. Biernacki, *Sur les fonctions multivalentes d'ordre p*, Paris Comptes Rendus, vol. 204 (1936), pp. 449–451.

LEMMA. *Every function $f(z)$ which is k-wise symmetric, regular, and multivalent of order p in the unit circle can be represented in the form*

$$f(z) = \{F(z^k)\}^{1/k}, \tag{2.1}$$

where $F(z)$ is regular and multivalent of order p in the unit circle. Conversely, if $F(z)$ is any function regular and multivalent of order p in the unit circle, and if $F(z)\neq 0$ for $0<|z|<1$, then $f(z)$ is also multivalent of order p and regular in the unit circle.

PROOF. We may assume $f(z)$ is given by (1.3) and define $F(z)$ to be

$$F(z) = z\left(\sum_{1+mk}^{\infty} a_n z^{(n-1)/k}\right)^k$$

which is regular within the unit circle. Thus (2.1) holds.

Since $f(z)$ is p-valent, then there is a value $te^{i\phi}$, $(t>0)$, so that $f(z_i)=te^{i\phi}$ for exactly p distinct values z_i, $(i=1, 2, \cdots, p)$. Moreover, since $f(z)$ is k-wise symmetric one has

$$f(z_i e^{2s\pi i/k}) = e^{2s\pi i/k}\cdot f(z_i) = te^{(2s\pi/k+\phi)i}$$

for $s=0, 1, 2, \cdots, (k-1)$. Thus $f^k(z_i')=t^k e^{k\phi i}$ for $z_i'=z_i e^{2s\pi i/k}$, that is for pk values z_i' of z. Thus $f^k(z)$, and consequently $F(z^k)$, are pk-valent. Hence $F(z)$ is p-valent. The converse may be proved similarly.

In the proof below we shall make use of an inequality of M. Cartwright* for multivalent functions $F(z)$ of order p in the unit circle. We assume here that $F(z)\neq 0$ for $0<|z|<1$ so that we have

$$|F(re^{i\theta})| < A(p)(1-r)^{-2p}. \tag{2.2}$$

The method of proof for (1.4) now is that of E. Landau† (for $p=1$, $k=2$) with but slight modifications to take care of $p>1$, $k\geqq 2$. Let $f(z)$ be defined as in (1.3), and $F(z)$ as in (2.1). Let $s=k+1$ and

$$\{F(z^s)\}^{1/s} = \sum_1^{\infty} c_n z^n = z^{1+mk} + \cdots,$$

$$\{F(z^{ks})\}^{1/ks} = \sum_1^{\infty} d_n z^n = z^{1+mk} + \cdots,$$

$$\sum_1^{\infty} a_n z^{ns} = \left(\sum_1^{\infty} d_n z^n\right)^s.$$

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* See M. Cartwright, *Some inequalities in the theory of functions*, Mathematische Annalen, vol. 111 (1935), pp. 98–118.

† See E. Landau, loc. cit. See also K. Joh and S. Takahashi, loc. cit.

By differentiation, we obtain

$$s\sum_1^\infty na_n z^{ns-1} = s\left(\sum_1^\infty c_n z^{nk}\right)\left(\sum_1^\infty nd_n z^{n-1}\right),$$

$$na_n = \sum_{k\mu+\nu=(k+1)n} c_\mu \nu d_\nu,$$

and we deduce the inequalities

$$n^2 \mid a_n \mid^2 < (k+1)n \sum_{\mu\leqq 2n} \mid c_\mu \mid^2 \cdot \sum_{\nu\leqq(k+1)n} \nu \mid d_\nu \mid^2,$$

$$\sum_1^m n \mid c_n \mid^2 r^{2n} \leqq p\left[\frac{A(p)}{(1 - r^{k+1})^{2p}}\right]^{2/(k+1)} < A_1(p, k)(1-r)^{-4p/(k+1)},$$

$$\sum_1^m n \mid d_n \mid^2 r^{2n} \leqq A_2(p, k)(1-r)^{-4p/ks}, \qquad s = k+1.$$

For $r = e^{-1/m}$, we then have for an arbitrary positive integer m

$$C_m = \sum_{n=1}^m n \mid c_n \mid^2 < A_3(p, k)m^{4p/(k+1)},$$

$$\sum_1^m n \mid d_n \mid^2 < A_4(p, k)m^{4p/ks}, \qquad s = k+1,$$

$$\sum_1^m \mid c_n \mid^2 = \sum_1^m \frac{C_n - C_{n-1}}{n} < A_5(p, k)m^{4p/(k+1)-1} \text{ for } k < 4p-1,$$

$$n^2 \mid a_n \mid^2 < (k+1)n\cdot A_5(p, k)(2n)^{4p/(k+1)-1}\cdot A_4(p, k)(\overline{k+1}\, n)^{4p/k(k+1)};$$

whence

$$\mid a_n \mid < A(p, k)n^{2p/k-1} \quad \text{for} \quad k < 4p-1.$$

For $k = 4p-1$ we may use the method of V. Levin* with the obvious modifications to take care of $p > 1$. This will give (1.4) for $k = 4p-1$. Thus (1.4) holds for $k < 4p$.

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* See V. Levin, loc. cit.