POSTULATES FOR BOOLEAN ALGEBRA IN TERMS OF TERNARY REJECTION*

BY ALBERT WHITEMAN†

1. *Introduction.* The operation of ternary rejection‡ in Boolean algebra is the operation ( ) given by $(abc) = a'b' + b'c' + c'a'$. In this paper, I shall present a set of postulates for Boolean algebras in which ternary rejection is taken as the only primitive idea, besides that of *class*. As a result, all the special Boolean elements are introduced with an elegance not possible in any other set known to the author. Thus, the *negative* of an element is defined in terms of the primitive ideas, and then *any* two contradictory elements are chosen to represent the *zero* element and the *universe* element of the resulting Boolean algebra.

We prove the *sufficiency* of the new postulates for Boolean algebra by deriving from them the well known Whitehead-Huntington set;§ the proof of *necessariness* consists in the converse derivation. Finally, we establish the *consistency* and *independence* of the postulates by furnishing proof-systems of the usual type.

2. *The New Postulates.* The new postulates have as undefined ideas a *class* K and a *ternary operation* ( ). The postulates are the propositions A_1-A_5 below. In Postulates A_3-A_5 the condition *if the elements involved and their indicated combinations belong to* K is to be understood.

POSTULATE A_1. *K contains at least two distinct elements.*

POSTULATE A_2. *If a, b, c are elements of K, (abc) is an element of K.*

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† Harrison Scholar in Mathematics, University of Pennsylvania.

‡ For other papers dealing with ternary operations see A. B. Kempe, *On the relation between the logical theory of classes and the geometrical theory of points*, Proceedings of the London Mathematical Society, (1), vol. 21 (1890), pp. 147–182; Orrin Frink, *The operations of Boolean algebras*, Annals of Mathematics, (2), vol. 27 (1925–1926), pp. 477–490; see also the bibliography at the end of Frink's paper.

§ See the Transactions of this Society, vol. 5 (1904), pp. 288–309.

POSTULATE A_3. $(abc)=(bca)$.

DEFINITION 1. $a'=(aaa)$.

POSTULATE A_4. $(a'bb')=a$.

POSTULATE A_5.* $[ab(cde)']=[(abc)'(abd)'e]$.

3. *Theorems*. We now list a number of theorems of the resulting algebra.

1. $a''=a$, *where* $a''=(a')'$.
2. $(aab)=a'$.
3. $(abc)=(acb)$.
4. $(abc)=(cba)=(bca)=(acb)=(cab)=(bac)$.
5. $[a'(abc)'(a'b'c')']=a$.
6. $[a(abc)'(ab'c')']=a'$.
7. $(abc)=[(abd)'(abd')'c]$.
8. $[d'(abc)'(a'b'c')']=d$.
9. *If* $(a'bc)=a$ *for all* a, *then* $c=b'$.
10. $(abc)'=(a'b'c')$.

4. *Proofs of the Preceding Theorems*.

PROOF OF 1. $a''=(a'''a'a'')=(a'a''a''')=a$, by A_4, A_3, A_4.

PROOF OF 2. $(aab)=[(abb')'(abb')'b]=[ab(b'b'b)']=(abb')=a'$, by A_4, 1, A_5, A_4, 1, A_4, 1.

PROOF OF 3.† Put $(acb)'=d$, $[c(abc)'b]'=e$. Then

$$(abc) = [ab(ccd)'] = [ab(cdc)'] = [(abc)'(abd)'c] = [c(abc)'(abd)'] = \{[c(abc)'a]'[c(abc)'b]'d\} = \{[c(abc)'a]'ed\} = \{[ac(abc)']'ed\} = \{[(aca)'(acb)'c]'ed\} = \{[(aac)'(acb)'c]'ed\} = \{[a(acb)'c]'ed\} = \{[ca(acb)']'ed\} = \{[(caa)'(cac)'b]'ed\} = \{[(aac)'(cac)'b]'ed\} = \{[a(cac)'b]'ed\} = \{[a(cca)'b]'ed\} = [(acb)'ed] = [d(acb)'e] = \{(acb)'(acb)'[c(abc)'b]'\} = (acb),$$

by 2, 1, A_3, A_5, A_3, A_5, A_3, A_5, A_3, 2, 1, A_3, A_5, A_3, 2, 1, A_3, 2, 1, A_3, 2, 1.

PROOF OF 4. By A_3, 3.

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* We shall use the symbols ( ), [ ], { } interchangeably to denote the fundamental ternary operation.

† I am indebted to H. S. Zuckerman for this proof.

In the rest of the proofs implicit use will be made of 1 and 4.

PROOF OF 5. $[a'(abc)'(a'b'c')'] = \{[a'(abc)'a']'[a'(abc)'b']'c'\} = \{a'[a'b'(abc)']'c'\} = \{a'[(a'b'a)'(a'b'b)'c]'c'\} = [a'c'(b'a'c)'] = [(a'c'a')'(a'c'c)'b'] = [a'(a'c'c)'b'] = (a'a'b') = a$, by A_5, 2, A_5, A_4, A_5, 2, A_4, 2.

PROOF OF 6. $[a(abc)'(ab'c')'] = \{[a(abc)'a]'[a(abc)'b']'c'\} = \{a[ab'(abc)']'c'\} = \{ac'[(ab'a)'(ab'b)'c]'\} = \{ac'[a(ab'b)'c]'\} = [ac'(aac)'] = (ac'a) = a'$, by A_5, 2, A_5, 2, A_4, 2, 2.

PROOF OF 7. $(abc) = [ab(dd'c)'] = [(abd)'(abd')'c]$, by A_4, A_5.

PROOF OF 8. Put $(abd)'=p$, $(abd')'=q$, $(a'b'd)'=r$, $(a'b'd')'=s$. Then

$$[d'(abc)'(a'b'c')'] = [d'(pqc)'(rsc')'] = \{[d'r(pqc)']'[d's(pqc)']'c'\}$$
$$= \{[(d'rp)'(d'rq)'c]'[(d'sp)'(d'sq)'c]'c'\},$$

by 7, A_5, A_5. But $(d'rq)=d$, $(d'sp)=d$ by 5, and $(d'sq)=d$ by 6. Hence the last expression reduces to

$$\{[cd'(d'rp)']'(cd'd')'c'\} = \{c'd'[cd'(d'rp)']'\} = [(c'd'c)'(c'd'd')'(d'rp)']$$
$$= [d'(c'd'd')'(d'rp)'] = [d'd'(d'rp)'] = d, \text{ by 2, } A_5, A_4, 2, 2.$$

PROOF OF 9. Suppose that for two fixed elements b, c, $(a'bc)=a$ for every element a. Then for $a=c$, $c=(c'bc)=b'$ by A_4.

PROOF OF 10. By 8, 9.

5. *Sufficiency and Necessariness of the Postulates.* The Whitehead-Huntington postulates leave undefined a *class* K and two *binary operations* $+$ and $\times$, and are the propositions Ia, Ib, $\cdots$, VI below. In postulates IIIa-IVb the condition *if the elements involved and their indicated combinations belong to* K is understood; in V the condition *if the elements* Z *and* u *of* IIa *and* IIb *exist and are unique* is understood.

POSTULATE Ia. *$a+b$ is in K whenever a and b are in K.*

POSTULATE Ib. *ab is in K whenever a and b are in K.*

POSTULATE IIa. *There is an element Z such that $a+Z=a$ for every element a.*

POSTULATE IIb. *There is an element u such that $au=a$ for every element a.*

POSTULATE IIIa. $a+b=b+a$.

POSTULATE IIIb. $ab=ba$.

POSTULATE IVa. $a+bc=(a+b)(a+c)$.

POSTULATE IVb. $a(b+c)=ab+ac$.

POSTULATE V. *For every element a there is an element $\bar{a}$ such that $a+\bar{a}=u$ and $a\bar{a}=Z$.*

POSTULATE VI. *There are at least two elements, a and b, in K such that $a\neq b$.*

We deduce Ia-VI from our postulates as follows:

Let u be *any* element in K. Then we may make the following definitions:

DEFINITION 2. $Z=u'$.

DEFINITION 3. $a+b=(abu)'$.

DEFINITION 4. $ab=(abZ)'$.

PROOF OF Ia. By Definition 3, A_2, Definition 1.

PROOF OF Ib. By Definition 4, A_2, Definition 1.

PROOF OF IIa. $a+Z=(aZu)'=(au'u)'=a$, by Definition 3, Definition 2, A_4.

PROOF OF IIb. $au=(auZ)'=(auu')'=a$, by Definition 4, Definition 2, A_4.

PROOF OF IIIa. $a+b=(abu)'=(bau)'=b+a$, by Definition 3.

PROOF OF IIIb. $ab=(abZ)'=(baZ)'=ba$, by Definition 4.

PROOF OF IVa. $a+bc=[a(bcZ)'u]'=[(abu)'(acu)'Z]'=(a+b)(a+c)$, by Definition 3, Definition 4, A_5, Definition 3, Definition 4.

PROOF OF IVb. $a(b+c)=[a(bcu)'Z]'=[(abZ)'(acZ)'u]'=ab+ac$, by Definition 3, Definition 4, A_5, Definition 3, Definition 4, PROOF OF V. $a + a' = (aa'u)' = u$, by Definition 3, A_4. $aa' = (aa'Z)' = Z$, by Definition 4, A_4. Hence $a' = \bar{a}$.

PROOF OF VI. By A_1.

In the converse derivation we may of course assume all the theorems of Boolean algebra, since they follow from the Whitehead-Huntington postulates. Postulates A_1-A_5 may then be verified without any difficulty after defining (abc) by $a'b' + b'c' + c'a'$.

6. *Relation between Ternary and Binary Boolean Algebra. Derivation of DeMorgan's Formula.* We first prove the fundamental relation $(abc) = a'b' + b'c' + c'a'$.

PROOF. $(abc) = (a'b'c')' = [(a'b'Z)'(a'b'Z')'c']'$
$= \{[Z'(a'b'Z)'Z]'(a'b'Z')'c'\}' = \{[Z'(a'b'Z)'Z]'[a'b'(ZZ'Z')']'c'\}'$
$= \{[Z'(a'b'Z)'Z]'[Z'(a'b'Z)'(a'b'Z')']'c'\}'$
$= \{Z'(a'b'Z)'[Z(a'b'Z')'c']'\}' = \{Z'(a'b'Z)'[(Zc'a')'(Zc'b')'Z']'\}'$
$= \{(a'b'Z)'[(b'c'Z)'(c'a'Z)'u]'u\}' = a'b' + b'c' + c'a'$, by 10, 7, A_4, 2, A_5, A_5, A_5, and Definitions 2, 3, and 4.

We next observe that 10 is a generalization of DeMorgan's formula. Indeed,

$$(a + b)' = (abu) = (a'b'u')' = (a'b'Z)' = a'b'.$$

7. *Consistency and Independence of the Postulates.*

The consistency of postulates A_1-A_5 is shown by the following example.

EXAMPLE 1.0. $K = 1, 2$.

$c = 1$

	1	2
1	2	2
2	2	1

$c = 2$

	1	2
1	2	1
2	1	1

The independence proofs follow.

EXAMPLE 1.1. $K = 1$.

$c = 1$

	1
1	1

EXAMPLE 1.2. $K=1, 2$.

$c=1$

	1	2
1	3	3
2	3	3

$c=2$

	1	2
1	3	3
2	3	3

EXAMPLE 1.3. $K=1, 2$.

$c=1$

	1	2
1	2	2
2	1	1

$c=2$

	1	2
1	2	1
2	1	1

Postulate A_3 fails for $a=2$, $b=c=1$.

EXAMPLE 1.4. $K=1, 2$.

$c=1$

	1	2
1	1	1
2	1	1

$c=2$

	1	2
1	1	1
2	1	1

Postulate A_4 fails for $a=2$, $b=c=1$.

EXAMPLE 1.5.* $K=1, 2$.

$c=1$

	1	2
1	1	2
2	2	1

$c=2$

	1	2
1	2	1
2	1	2

Postulate A_5 fails for $a=c=1$, $b=d=e=2$.

THE UNIVERSITY OF PENNSYLVANIA

* The following alternate proof-systems, due in part to J. C. C. McKinsey, are of interest. Ex. 1.0. $K=1, 2$; $(abc)=[(a+b+c)(a+b+c+1)/2](\text{mod } 2)$. Ex. 1.1. $K=1$; $(abc)=1$. Ex. 1.2. $K=1, 2$; $(abc)=3$. Ex. 1.3. $K=1, 2$; $(abc)=a$. Ex. 1.4. $K=1, 2$; $(abc)=1$. Ex. 1.5. $K=1, 2$; $(abc)=[a+b+c](\text{mod } 2)$. A_5 fails for $a=c=d=e=1$, $b=2$.