ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

1. Professor C. R. Adams and Mr. A. P. Morse: *On the space* (BV).

The authors consider the space (BV) of functions $x(t)$ of bounded variation on $0 \leqq t \leqq 1$, metrized thus: $(x, y) = \int |x(t) - y(t)| dt + |T(x) - T(y)|$ (see Adams, Transactions of this Society, November, 1936). The following semi-continuity property of the functional $T(x)$ [also enjoyed by $L(x)$, the Peano length of $x(t)$] is basic: $x_n(t) \rightarrow x(t)$ on a dense set, with $x(t)$ continuous at the end-points and having an external saltus at no interior point, implies $\underline{\lim}\ T(x_n) \geqq T(x)$. Several questions naturally raised by the paper cited may now be answered, and it can be shown that the set $(DBV) =$ (the discontinuous elements of (BV)) is the sum of non-dense closed sets in (BV); whence $(CBV) =$ (the continuous functions in (BV)) is a residual set in (BV). Likewise $(CBV - CS)$, where (CS) $=$ (the continuous singular functions), is the sum of non-dense closed sets in (CBV); and (CS) is a residual set in (CBV). Schauder's base for (C) is a base also for (CBV); and Haar's system of orthogonal functions, augmented by the characteristic function of the set $\{1\}$, provides a base for (BV). (Received November 30, 1936.)

2. Professor V. W. Adkisson and Dr. Saunders MacLane: *On planar graphs whose homeomorphisms can all be extended.*

This paper considers the non-separable planar graphs such that, regardless of how the graph is mapped on the sphere, all the homeomorphisms of the graph into itself can be extended to the sphere. The idea of proper components is utilized. A graph G is split into two proper components, H_1 and H_2, if $G = H_1 + H_2$, if neither H_1 nor H_2 is a single edge, and if H_1 and H_2 have in common only two vertices (the "end" vertices). A proper component is minimal if no split of the graph gives a component which is a proper subset of H_1. If, regardless of how G is mapped on the sphere, all its homeomorphisms are extendible then either (1) G cannot be split into proper components, (2) G has no non-trivial homeomorphisms, (3) G contains only two minimal components of a specified type, (4) G has just one non-trivial homeomorphism of order two, and this homeomorphism merely exchanges the end vertices of each minimal component (the vertices the component has in common with the rest of G), with no rotation of the element with respect to these vertices. Conversely, if any one of the above conditions is satisfied, then all homeomorphisms of G into itself are extendible regardless of how G is mapped. (Received November 27, 1936.)

3. Professor W. L. Ayres: *On transformations having periodic properties.*

Using the Whyburn cyclic element theory a study is made of continuous transformations of compact locally connected spaces into themselves which are periodic, point-wise periodic, almost periodic, or point-wise almost periodic. While these four types of transformations are quite different, it is found that they have identical properties with regard to the cyclic elements. Among the results are the following: the set of cyclic elements which are invariant in the large is non-vacuous, closed, and connected. If A and B are cyclic elements such that $f(A)=B$, then the cyclic chain from A to B contains one and only one invariant cyclic element. (Received November 30, 1936.)

4. Mr. Garrett Birkhoff: *Analytical groups.*

A class of abstractly defined "analytical groups" is defined, embracing Lie groups and the infinite continuous groups studied by Delsarte (but not those studied by Lie and Cartan). These groups are found to possess many properties of Lie groups, namely, one-one representability by canonical parameters, continuous homeomorphism with an adjoint, existence of a Lie algebra, correspondence between Lie subalgebras and subgroups, invariant Lie subalgebras and invariant subgroups, analytical function of composition under canonical parameters, and so on. Also, it is pointed out that the invariants of Lie groups are all topologico-algebraic. Further, an infinitesimal calculus for analytical groups is developed, which extends the Riemann-Volterra and the Lebesgue-Schlesinger theories for matrices to non-linear groups and to infinite continuous groups. Various results ignored or overlooked in the case of matrices are also given. (Received November 18, 1936.)

5. Mr. Garrett Birkhoff: *On the integral calculus of operators.*

An answer is given to the following question. Let $X(t)=\sum_{h=1}^{r}\rho_h(t)\cdot X^h$ be a variable linear combination of several fixed operators X^h operating over a specified period of time. What single fixed operator Z, operating constantly over the same period, will produce the same net effect? The computation of Z involves only the addition, multiplication, quadrature, and differentiation of the $\rho_h(t)$, and the taking of successive Poisson brackets of the X^h. (Received November 18, 1936.)

6. Professor H. E. Bray: *On the roots of the derivative of a polynomial.*

The author proves the following theorem: If $z_1, z_2, \cdots, z_n$, are the roots of a polynomial $P(z)$, and if ξ is a root of the derivative $P'(z)$ but not of $P(z)$, and if $|z_1|<|z_2|<\cdots<|z_n|$ are held fixed, then $|\xi|$ attains its greatest possible value when and only when $z_1, z_2, \cdots, z_n$, all have the same amplitude. The properties of the cubic curve (in polar coördinates) $r=a \sin(\alpha-\theta/2)/\sin(\theta/2)$ provide the essential facts in the proof. (Received November 30, 1936.)

7. Dr. E. A. Cameron and Professor J. W. Lasley: *On certain loci associated with a plane curve.*

The equations of conics of various orders of contact are obtained in both local and general systems of coordinates. A study is made of properties and relations of these osculants and other connected loci at the point. As the point moves along the curve associated loci are generated by parts of penosculating conics. The equations of a number of such loci have been obtained and their properties established. (Received November 27, 1936.)

8. Dr. R. H. Cameron: *Quotients of almost periodic functions.*

This paper deals with solutions $x(t)$ of the equation $f(t)x(t) \equiv g(t)$, in which $f(t)$ and $g(t)$ are uniformly almost periodic functions and $g(t)$ has no translation transform which vanishes throughout an interval. It is shown that a necessary and sufficient condition that $x(t)$ be uniformly almost periodic is that it be bounded and uniformly continuous. (Received November 23, 1936.)

9. Dr. R. H. Cameron and Dr. W. T. Martin: *Analytic continuation of diagonals and Hadamard compositions of multiple power series.*

The authors consider an analytic function $A(x, y) = \sum a_{mn}x^m y^n$ and its star S, a region which consists of all points (x_0, y_0) such that A is regular for $x = \rho x_0$, $y = \sigma y_0$ $(0 \leqq \rho \leqq 1, 0 \leqq \sigma \leqq 1)$. They define a *contracted* star S^* which is the region in the z-plane consisting of all points z for which there exists a continuous positive periodic function $r(\theta)$ such that all of the points $(z_n(\theta)e^{i\theta}, e^{-i\theta}/n(\theta))$ are in S. It is shown that the *diagonal function* $A^*(w) = \sum a_{mm}w^m$ *is regular in* S^*. *When* $A(x, y) = B(x)C(y)$ *this result gives the classical Hadamard result.* By iteration of this theorem it is shown that the diagonal function $A^{**}(z, w) = \sum a_{mnmn}z^m w^n$ of a function $A(x, y, u, v)$ is analytic in a star-shaped region characterized in terms of the star of $A(x, y, u, v)$. In particular, if $A(x, y, u, v) = B(x, u)C(y, v)$ then the diagonal function is the two-variable analogue of the Hadamard composition, namely $\sum b_{mn}c_{mn}z^m w^n$. By continued iteration, these results can be carried over to any number of variables. (Received November 27, 1936.)

10. Professor Leonard Carlitz: *A class of polynomials.*

For $\psi(t)$ as defined in the Duke Mathematical Journal (vol. 1 (1935), p. 148) the polynomials $\omega_M(t)$ are defined by $\psi(Mt) = \omega_M(\psi(t))$. In turn polynomials $W_M(t)$ are defined by $\omega_M = \prod W_A$, the product extending over all A dividing M, where M is an arbitrary polynomial in $GF(p^n)$. In this paper ω_M is defined algebraically without the use of the function $\psi(t)$, and various properties of W_M deduced. The polynomials W_M are seen to be analogous to the cyclotomic polynomials. (Received November 18, 1936.)

11. Professor Leonard Carlitz: *A criterion for certain higher congruences.*

The congruences in question are $\sum_{i=0}^{s}(-1)^i[s^s - i]^{p^i}u^{p^i} \equiv M^{p^s} \pmod{P}$, where M is an arbitrary polynomial in a single indeterminate x with coefficients in a finite field, and P is an irreducible polynomial of this kind; the coefficients $[s^s-i]$ are certain polynomials in x defined explicitly in the Duke Mathematical Journal (vol. 1 (1935), p. 141). If P is of degree $k>s$, it is shown that the congruence is solvable if and only if $P'M$ (where P' denotes the derivative) is congruent (mod P) to a polynomial of degree $<k-s$. Thus this theorem is a direct generalization of a known theorem (the case $s=1$). (Received November 30, 1936.)

12. Professor Leonard Carlitz: *Some formulas for factorable polynomials in several indeterminates.*

A factorable polynomial in $GF(p^n)$ is defined by the product $\Pi_j(\alpha_{j0}+\alpha_{j1}x_1 + \cdots +\alpha_{jk}x_k)$, where the α_{ji} are in some $GF(p^{nm})$ (see Duke Mathematical Journal, vol. 2 (1936), No. 4). In this note some formulas for the case $k=1$ (ordinary polynomials in a single indeterminate) are extended to the case $k>1$. (Received November 18, 1936.)

13. Professor J. M. Clarkson: *Cremona transformations of plane curves expressed by third-order differential equations.*

Given three independent solutions x_1, x_2, x_3 of the equation (1) $x'''+3p_1x''+3p_2x'+p_3x=0$, where x_i are functions of a parameter t and where primes indicate differentiation with respect to t, if these x_i be taken as the homogeneous coordinates of a point in the plane, then as t varies, the x_i trace a curve expressed by (1). An investigation is made of the behavior of the coefficients p_j when a Cremona transformation (2) $y_i=y_i(x_1, x_2, x_3)$ is performed on x_i. (Received November 25, 1936.)

14. Dr. W. S. Claytor: *Peanian continua not imbeddable in a spherical surface.*

In a primitive skew curve K let a and b be vertices which bound one of its edges ab, and let C denote the curve $\overline{K-ab}$. If $C_n(=t(C))$, is any topological image of C, then set $a_n=t(a)$ and $b_n=t(b)$. In 3-space take (1) an infinite sequence of curves $\{C_n\}$ each of which is homeomorphic with C, (2) an infinite sequence of arcs $\{b_na_{n+1}\}$, and (3) a single arc PQ; such that (1) $C_iC_j=(b_ia_{i+1})(b_ja_{j+1})=C_i\cdot PQ=(b_ia_{i+1})\cdot PQ=0$, for $i\geqq 1$ and $i\neq j$; (2) $(b_ia_{i+1})\cdot\sum_{n=1}^{\infty}C_n=(b_ia_{i+1})\cdot(C_i+C_j)=b_i+a_{i+1}$, $(i\geqq 1)$; (3) $\lim_{n=\infty}(C_n+b_na_{n+1})=P$. Then set $\Delta=\sum_{n=1}^{\infty}(C_n+b_na_{n+1})+PQ$. It is readily seen (as was shown by Kuratowski, Fundamenta Mathematicae, vol. 15, p. 272, footnote 2), that Δ is a curve which does *not* have a homeomorphic image in the plane. Furthermore, since there are just *two* primitive skew curves (topologically speaking), it follows that Δ is homeomorphic with one or the other of two curves which are denoted by Δ_1 and Δ_2. The following theorem is proved: a Peanian continuum which is not homeomorphic with a subset of a spherical surface necessarily contains either a primitive skew curve or a topological image of one of the curves Δ_1 and Δ_2. For the definition of a primitive skew curve, see the paper by this author in the Annals of Mathematics (vol. 35 (1934), No. 4). (Received November 28, 1936.)

15. Professor L. W. Cohen: *Transformations on spaces with denumerable basis.*

Conditions that a linear Tx on a Banach space with denumerable basis to another such space be completely continuous and that the matrix of $(I+T)x$ have an absolutely convergent determinant are given. In the case of Tx on the space l_1 with $\|x\| = \sum |x_i|$ to the space l_0 with $\|x\| = \sup |x_i|$, $\lim x_i = 0$, the condition for complete continuity becomes $\lim_n \sup_i \sup_{m>n} |a_{mi}| = 0$, where $\|a_{mi}\|$ is the matrix for T. In the case of $(I+T)x$ on l_0 to l_0, the conditions become (1) $\lim_m \sum_i |a_{mi}| = 0$, (2) $\sum_i \sup_m |a_{mi}| < +\infty$. (Received December 1, 1936.)

16. Dr. E. G. H. Comfort: *On the preservation of Hölder properties of initial conditions in the solutions of wave equations.*

A function is said to have a Hölder property if it satisfies a Hölder condition. Kellogg has shown (Transactions of this Society, vol. 33 (1931), p. 486–510) that, for harmonic functions in a sphere, Hölder properties of the boundary values (or of their derivatives) imply Hölder properties (same order) of the harmonic function itself (or of the corresponding derivatives) in the closed sphere. The same holds true for the particular wave equation $u_{tt} - u_{xx} = 0$, that is, Hölder properties of the initial conditions (or of their derivatives) are preserved with the same order in the solution (or in the corresponding derivatives of the solution). This paper shows that such is not the case for the wave equation $u_{tt} - u_{xx} - u_{yy} = 0$. If the Hölder properties of the initial conditions (or of their derivatives) have an order $\alpha > 1/2$, then the solution (or its corresponding derivatives) will have a Hölder property of order $\alpha - 1/2$. Corresponding results are shown to hold for the wave equations $u_{tt} - u_{xx} - u_{yy} - Ku = 0$, and $u_{tt} - u_{xx} - u_{yy} = \phi(x, y, t)$. (Received November 20, 1936.)

17. Professor H. B. Curry: *On the use of dots as brackets in logical expressions.*

The author suggests a slight modification of the Peanese convention which enables one to represent conveniently chains of indefinite length, such as the formula $a_1 \supset (a_2 \supset (\cdots \supset (a_n \supset b) \cdots))$ (even when the a_i are themselves complex), without doing violence to our ordinary algebraic usage. (Received November 21, 1936.)

18. Professor E. L. Dodd: *Some internal and external means arising from the location of frequency distributions.*

The problem of location for frequency functions as introduced by R. A. Fisher leads to generalized means or substitutive means which may be internal or external. A substitutive mean of observations $x_1, x_2, \cdots, x_n$, relative to a function F, is a solution M of $F(x_1, x_2, \cdots, x_n) = F(M, M, \cdots, M)$, as defined by O. Chisini. The external means occur rather naturally with bimodal distributions. Under certain simple conditions the Fisher likelihood process leads to exterior means with maximum likelihood and to interior means with minimum likelihood. For the problems here considered, likelihood is equivalent to the *a posteriori* probability, with the *a priori* probability taken as constant.

With this interpretation, the problem is that of finding the "most probable" location for frequency distributions. (Received November 20, 1936.)

19. Dr. F. G. Dressel: *A note on Fredholm-Stieltjes integral equations.*

If $K(x, y)$ is absolutely continuous with respect to a monotone function $g(y)$ for all x, then the paper shows that $f(x) = m(x) + \lambda \int_0^1 f(y) dK(x, y)$ is equivalent to an ordinary Fredholm equation. (Received November 24, 1936.)

20. Dr. D. M. Dribin (National Research Fellow): *Quartic fields with the symmetric group.*

In the present paper the possible types of groups of decomposition, inertia, and ramification that are allowed by the normal extension of a quartic field over the rational numbers having the symmetric group are studied, and those types which may exist are determined. These normal fields N are constructed by forming the direct product of relatively quadratic fields Λ_1 and Λ_2 over a normal sextic field B and insuring that the conductors of Λ_1 and Λ_2 be divisible by the proper powers of certain prime ideals in B. This investigation is closely connected with a paper of Hasse on non-normal cubic fields; problems kindred to that under consideration have been studied by Porusch (metabelian fields of quite special type) and by Rosenblüth (the "quaternionic" field). (Received November 30, 1936.)

21. Dr. Nelson Dunford: *Integration of vector-valued functions.* Preliminary report.

A function $f(P)$ with values in a Banach space X is said to be integrable and belong to the class S_p in case $Tf(P)$ is in L_p for every T in $\bar{X}$. For such functions $\int_e Tf(P)dm$ is continuous in T, and thus defines a point in $\bar{\bar{X}}$. This point is taken as the value of $\int_e f(P)dm$. A similar definition holds for the integral of a numerical function with respect to an abstract-valued measure function. The present paper discusses the properties and uses of the above integrals. (Received November 30, 1936.)

22. Dr. Nelson Dunford: *Linear transformations of sequences.*

A necessary and sufficient condition for a function $y = Tx$ to be linear and continuous on l to $l_p (1 \leqq p < \infty)$ is that it can be given in the form $\eta_i = \sum_{j=1}^{\infty} a_{ij} \xi_j$ where $\sup_j \{\sum_{i=1}^{\infty} |a_{ij}|^p\}^{1/p} < \infty$. This last constant is the evaluation of $|T|$. For Banach spaces with a base $\{\phi_i\}$ a necessary and sufficient condition for the compactness of a set E is $x = \sum_{i=1}^{\infty} a_i \phi_i$ uniformly with respect to x in E. This yields as necessary and sufficient conditions for the complete continuity of $y = Tx$ on l to l_p the above condition as well as the condition $\lim_{n=\infty} \sup_j \{\sum_{i=n}^{\infty} |a_{ij}|^p\}^{1/p} = 0$. Sufficient conditions for complete continuity of $y = Tx$ on l_p to l_q are also given. (Received November 30, 1936.)

23. Professor L. A. Dye: *A transformation associated with the trisecants of a rational twisted quintic curve.*

This transformation is generated by the use of a (1, 1) correspondence between a pencil of ruled cubic surfaces $|F_3|$ and the trisecants of a rational twisted curve C_5. The trisecants of C_5 are the simple directrices of $|F_3|$ whose base is the C_5 together with a double line which is the quadrisecant of C_5. The most interesting feature of the transformation is the existence of two ruled surfaces whose generators are fundamental lines of the second species of the transformation. One of these surfaces is the ruled surface of trisecants of C_5 and it is not a principal surface of the transformation. The other surface, which is a principal surface, is generated by the tangent lines to C_5 at the points where it cuts the trisecants which are simple directrices of $|F_3|$. (Received November 17, 1936.)

24. Dr. M. M. Flood: *Column normal matric polynomials.*

The principal questions answered in this paper are: (i) What is the greatest lower bound for $d[PX]$ when P is a given matric polynomial, X is a matric polynomial of prescribed rank, and $d[PX]$ is the degree of the product PX? (ii) How can the degree invariants of the column vector space of $C_r[P]$ be expressed in terms of the degree invariants of the column vector space of P, where P is a given matric polynomial and $C_r[P]$ is its rth compound? (iii) How can the problem of finding the matric roots of a matric polynomial P be reduced to the consideration of regular matric polynomials? (Received November 25, 1936.)

25. Dr. H. H. Goldstine: *The calculus of variations in general analysis.* I.

Making use of E. H. Moore's principle of generalization by abstraction, the author was led to attempt to find the general theory underlying both the calculus of variations and the theory of the minima of functions of several variables. In this paper the author has made a first step in this direction by considering the problem of the minima of functions defined on regions of the space of modular functions. The theory developed here includes as special cases the simple fixed end-point problem of the calculus of variations in which the admissible arcs have Lebesgue square integrable derivatives almost everywhere, the theory of the minima of functions of a finite number of variables, and a similar theory for functions of Hilbert sequences. By combining the method of his thesis with that of the second form of general analysis, the author obtains analogs of the multiplier rule, the Clebsch condition, and the Jacobi-Mayer condition; moreover when the end condition is suitably restricted, sufficient conditions are obtained by expansion methods. The analog of the Jacobi condition is especially interesting because, in the case of the calculus of variations, the author has shown it is a stronger condition than the one ordinarily stated. (Received November 23, 1936.)

26. Mr. C. H. Graves: *On certain surfaces in five-dimensional space.*

In this paper a study is made of certain surfaces in five-dimensional space sustaining families of tritangent curves. A tritangent curve is defined as a curve having the property that the ambient space of three consecutive tangent planes of the surface along the curve is an S_4. After studying the differential geometry of surfaces in S_5, the surface of Veronese is defined and a new characteristic property given: If a surface of S_5 with $S(2, 0)=S_5$ possesses for each of its points three distinct tritangent curves which are also quasi-asymptotics $\gamma_{1,3}$ it is necessarily a surface of Veronese. The conditions on the coefficients in the system of differential equations defining a surface which are necessary for the surface to sustain tritangent curves are developed in a general manner. The particular case of a surface sustaining five tritangent curves is developed in more detail. The problem of the existence of surfaces other than the surface of Veronese on which there exist families of tritangent curves is considered and three special surfaces are defined, one sustaining three tritangent curves through each point, another sustaining four, and the third surface sustaining five. (Received November 18, 1936.)

27. Professor T. R. Hollcroft: *Contacts of algebraic plane curves.*

Aside from the multiple contacts of conic and cubic, the contacts of algebraic plane curves have been studied but little. Since the proof of a theorem by B. Segre (B. Segre, *Esistenza e dimensione di sistemi continui di curve piane algebriche con dati caratteri*, Rendiconti Accademia dei Lincei, (6), vol. 10 (1929), pp. 36–37, Theorem 2^0) relating the number of contacts of two curves to the number of cusps of a third curve whose order is the sum of the orders of the first two, the subject of contacts of two curves has taken on new and greater significance. In this paper limits and upper bounds to the number of contacts of two plane curves are obtained in certain cases. (Received November 30, 1936.)

28. Dr. Charles Hopkins: *Concerning the structure and representations of a certain class of finite rings.*

The author considers a ring O with the following characteristics: the number of elements is finite; every left-hand nilfactor is nilpotent; O contains an element which is not a left-hand nilfactor and also an element which is not a right-hand nilfactor. Then O is a hypercomplex system over a Galois field F (of order p^f) and is the direct sum of two subalgebras F^* and R, where F^* is simply isomorphic with F and R is nilpotent. The elements of O which are congruent modulo R to the principal unit of O constitute under multiplication a group Γ of order p^{nf}, where n is the rank of R over F. The connection between the structure of Γ and the ideal structure of R is investigated; the number of non-equivalent cyclic O-moduli is obtained in terms of certain invariants of R. The group-ring of a finite p-group G over a finite field of characteristic p is a special example of the ring O; accordingly one is able to obtain systematically the known results concerning representations of G by linear transformations over F. (Received November 30, 1936.)

29. Dr. C. C. Hurd: *Asymptotic theory of linear differential equations singular in a parameter and in the variable of differentiation.*

This paper considers linear differential equations whose coefficients are series in negative integral powers of a complex parameter λ and the complex variable x, with possibly a finite number of positive powers of λ and x present. Under the hypothesis that no root of multiplicity greater than one of the characteristic equation satisfies a certain "auxiliary equation," the existence of a full set of formal solutions is demonstrated. The method of iterations, developed by Trjitzinsky for differential equations, is used to establish the asymptotic character of the solutions. (Received November 23, 1936.)

30. Mr. L. P. Hutchinson: *The Lagrange multiplier theorem for normed vector space.*

On XY, the composite of two linear normed spaces, $f(x, y)$ is of class C' to the space of real numbers and $F(x, y)$ is of class C' to Y. Subject to $F(x, y)=0$, $df=0$ at (x_0, y_0). Then at this point there is a linear functional L such that $LdF=df$. The basis of the proof lies in the Hildebrandt-Graves implicit function theorem and the transformation $L_x=L_yT$ on the spaces of linear functionals on X, Y associated with the transformation $Tx=y$. (Received December 1, 1936.)

31. Professor Dunham Jackson: *Orthogonal polynomials in three variables.*

The extension of the theory of orthogonal polynomials from two variables to three naturally proceeds to a considerable extent automatically. Nevertheless the increase in the number of dimensions gives rise to some new questions of interest, which are discussed in the present paper. As possibilities calling for separate consideration, the domain of orthogonality may be a three-dimensional region, or an algebraic surface, or a non-algebraic surface, or an algebraic curve, or a non-algebraic curve on an algebraic surface, or a curve which does not lie on an algebraic surface. (Received November 30, 1936.)

32. Professor Dunham Jackson: *Polynomial approximation on a curve of the fourth degree.*

In another paper (abstract 42-9-335) the writer has discussed the formal properties of polynomials in two real variables which are orthogonal on a curve in the plane of the variables. In the simplest cases, when the curve is a line segment or a circle, the resulting series developments reduce essentially to Legendre and Fourier series respectively, so that in those cases new problems of convergence do not arise. This paper is concerned with questions of the convergence of polynomial approximation on a particular curve of the fourth degree, for which the results are new in substance as well as in form. (Received November 30, 1936.)

33. Professor Dunham Jackson: *Problems of closest approximation on a two-dimensional region.*

In previous articles (this Bulletin, vol. 39 (1933), pp. 889–906; Transactions of this Society, vol. 40 (1936), pp. 225–251) theorems have been presented specifying upper bounds for the magnitude of polynomials and trigonometric sums normalized with respect to a given weight function, and corresponding inequalities relating to the approximate representation of functions of a single real variable. Results of similar character are now derived for regions of the (xy) plane. While the extension is immediate in some of its aspects, it also introduces new considerations which give the problem in two variables a degree of interest for its own sake. (Received November 30, 1936.)

34. Professor Fritz John: *Polar correspondence with respect to a convex region.*

Let R be a convex region in n-dimensional projective space. A 1-1 correspondence between the points of R and the hyperplanes outside R will be denoted as a polar correspondence, P. C. A P. C. is called positive if a point and its polar plane are not separated by any other point and its polar plane. It is proved that a positive P. C. is always continuous. A P. C. is called symmetric if in the neighborhood of any point it can be approximated by an ordinary P. C. with respect to a quadric. Every positive mass distribution on R gives rise to a positive symmetric P. C. by choosing as pole of hyperplane p the center of mass P of the distribution with p as plane at infinity. The P. C. thus generated is in the neighborhood of P approximately given by the P. C. with respect to Legendre's ellipsoid of inertia of the mass distribution. (Received November 27, 1936.)

35. Professor Edward Kasner: *General trihorn geometry.*

The configuration formed by three curves starting from a common point in a common direction is a fundamental object in conformal geometry and is termed a *trihorn.* The three horn angles in the trihorn have measures M_{12}, M_{23}, M_{31} as defined in the Proceedings of the National Academy, May, 1936. Any two of the curves determine a linear pencil of curves (wide open position, Schwarzian symmetry and conformal bi-section). Any two linear pencils have a new unique conformal invariant which is defined as the "angle" of the two pencils. Thus we have three new angles α_1, α_2, α_3. The angles are subject to an identity—their product is unity. The three measures, however, are subject only to an inequality relation. The complete geometry of trihorns, which may be called *trihornometry,* studies the interrelations, equalities and inequalities, between the six invariants. In the study of this geometry there is introduced an auxiliary plane with cartesian coordinates $x=\gamma$, $y=d\gamma/ds$, where γ denotes curvature and s arc length. The induced group in this plane is $X=mx+h$, $Y=m^2y+k$. The metric in this plane is defined by the calculus of variation problem $\int dx^2/dy=\int(1/y')dx=$minimum. The medians of a triangle are concurrent so the centroid exists; but the altitudes of a triangle are not concurrent so the orthocenter does not exist. Many other theorems in triangle geometry are extended to conformal theory. (Received November 27, 1936.)

36. Professor Edward Kasner: *Special trihorns.*

A trihorn has three measures (sides) and three angles. It is shown that neither equilateral nor equiangular trihorns exist. If two sides are equal it does not follow that the opposite angles are equal—in fact this never can happen. Isosceles right triangles exist of two distinct types, one with angles $2, 3/2, 1/3$, the other with angles -2, $1/2$, -1. Triangles can exist with two right angles. Perpendicularity (transversality) is defined by the angle $1/2$ or 2. If two lines are parallel the perpendicular distance from one to the other and the anti-perpendicular distance are in the ratio $-8:1$. The minimum distance is defined by the perpendicular. (Received November 27, 1936.)

37. Dr. P. W. Ketchum: *On the possible behavior of an analytic function at a set of isolated points.*

Several theorems which are related to and in some cases generalizations of the Mittag-Leffler and Weierstrass factor theorems are obtained for an arbitrary isolated set of points $\{a_n\}$. The problem in general terms is to construct, if possible, an analytic function $A(z)$ with certain preassigned properties in the neighborhood of each point a_n. The only permitted singularities of $A(z)$, other than those preassigned, will be at the limit points of $\{a_n\}$. As an example, we construct, for an arbitrary isolated set $\{a_n\}$, a function $A(z)$ with a simple zero at each point a_n and whose level curves $|A(z)|=\eta$ will, for every sufficiently small η, contain separate branches about each point a_n. Applications are made to the problem of simultaneously expanding a set of functions $\{f_n(z)\}$, where $f_n(z)$ is analytic at a_n, in the same series $\Sigma c_m A_m(z)$, the series to converge uniformly to $f_n(z)$ near a_n and the c's to be uniquely determined. (Received November 27, 1936.)

38. Dr. V. V. Latshaw: *On second order adjoint difference systems.*

A definition of second order linear adjoint difference systems is formulated. Explicit conditions for self-adjointness are detailed and two systems fulfilling these conditions are included. (Received November 23, 1936.)

39. Dr. A. N. Lowan: *On the operational determination of Green's functions in the theory of heat conduction.*

A Green's function $G(x, y, z; \xi, \eta, \zeta; t, \tau)$ in the theory of heat conduction for a volume V bounded by a surface S is defined as a solution of the equation (A): $\partial u/\partial t - k\Delta u=0$ for which (1) $\alpha\partial u/\partial n+\beta u=0$ over S ($\partial u/\partial n$ signifying differentiation along the normal), and which (2) for $t=\tau$ vanishes everywhere except at some point $P(\xi, \eta, \zeta)$ at which $u=\infty$ in such a manner that (3) the volume integral $\int\int\int u(x, y, z)dxdydz=1$. The Green's function G is obtained by the superposition of a "point source" solution v of (A) satisfying (2) and (3) and a solution w of (A) vanishing identically for $t=\tau$ and satisfying the further condition $(\alpha\partial/\partial x+p)(v+w)=0$ over S. The object of the present paper is the operational determination of the Green's function. This is obtained by constructing the Laplace transform v^*+w^* of G and then subjecting the now known function w^* to the inverse Laplace transformation. The last task is accomplished with the aid of certain standard theorems of the operational calculus. The method is illustrated for the case of a semi-infinite solid, but may be suitably extended to the case of a sphere, a cylinder, and so on. (Received November 23, 1936.)

40. Dr. Saunders MacLane: *A structural characterization of planar combinatorial graphs.*

This paper develops a criterion that a combinatorial graph be planar, which is stated explicitly in terms of natural structural properties of graphs. A *split* of a cyclicly connected graph G is a representation $G=H_1+H_2$, in which the two subgraphs H_1 and H_2 have no arcs and just two vertices p and q in common. The "blocks" of this split are obtained by adding to H_1 and H_2 respectively two new edges with the ends p and q. These two blocks may be split in their turn until we obtain unsplittable blocks or "atoms." These atoms are essentially unique. It is shown that the original graph is planar if and only if its atoms are planar. Furthermore an atom A is planar if and only if the circuits of A which do not separate A can be taken as the domain boundaries of a map of the atom A. This is possible when these circuits contain every edge exactly twice and have no other linear relation modulo 2 (S. MacLane, abstract 40-5-195). A graph is planar if its atoms have this property. (Received November 27, 1936.)

41. Professor J. D. Mancill: *On problems of the calculus of variations for which transversality is equivalent to orthogonality.*

It is well known that if the transversality condition for the minimum of the integral $\int f(x, y, y')dx$ shall be equivalent to orthogonality it is necessary and sufficient that f have the form $g(x, y)(1+y'^2)^{1/2}$. The purpose of this note is to present a rather simple proof of the analogous property for parametric problems in space of any number of dimensions. The same procedure is used to determine the integrand function $F(x, y, x', y')$, that is, in the plane case, for which the tangent of the angle between the direction determined by (x', y') and the corresponding transversal direction at each point is some prescribed function α of the coordinates of the point. The function F in this case is of the form $F(x, y, x', y')=G(x, y)(x'^2+y'^2)^{1/2}\cdot\exp\left[-\arc\tan (y'/x')/\alpha(x, y)\right]$. (Received November 30, 1936.)

42. Dr. A. J. Maria: *Equilibrium point of Green's function for a spherical shell.*

Let O be the center of the spheres of radii R_1 and R_2, $R_1<R_2$, bounding the shell Σ; let the pole of the Green's function for Σ be at a distance r_0 and the corresponding equilibrium point at a distance r from O. The principal result of this article is that $R_1<\alpha<r<\beta<R_2$ when $R_1<r_0<R_2$. (Received November 30, 1936.)

43. Professor C. N. Moore: *On the regularity of methods of summation of multiple series.*

By the use of various general theorems on convergence factors, recently obtained, for convergent and restrictedly convergent multiple series, necessary and sufficient conditions are derived for the regularity of definitions of summability in the case of multiple series. Criteria are obtained both for convergence factor definitions and sequence-transformation definitions, and the case of restricted summability is dealt with as well as the case of summability in the general sense. (Received November 30, 1936.)

44. Professor R. L. Moore: *A characterization of a compact continuum with no essential continuum of condensation.*

It is proved that in order that the compact continuum M should have no essential continuum of condensation it is necessary and sufficient that for every two points X and Y of M there exist an uncountable collection G of mutually exclusive point sets each consisting of either one or two points of M and each separating X from Y in M. (Received December 1, 1936.)

45. Professor R. L. Moore: *Concerning the open subsets of a plane continuum.*

W. T. Reid has shown (this Bulletin, vol. 41 (1935), pp. 684–688) that if K is a proper subcontinuum of a plane continuum M then K contains a limit point of some component of $M-K$. It is shown that (1) the above proposition does not remain true if the word "subcontinuum" is replaced by "closed subset" and (2) if, in a plane, K is a closed point set and G is a countable set of mutually exclusive continua and G^* (the sum of all the continua of the set G) has no point in common with K and G^*+K is a continuum, then every element of G is a component of G^*. (Received December 1, 1936.)

46. Professor Marston Morse: *Homotopic extremals.*

This paper continues with the systematic development of ideas presented at various times during the last two years. We are concerned with an abstract variational theory based on a study of a function F lower semi-continuous on an abstract space M. The theory has two principal parts: (I) the analysis of the distribution of critical values by group theoretic means, involving indexed subsets of additive, abelian, operator groups, and (II) a local topological definition of a critical point. We are here concerned with (II); and, of four different types of definitions of critical points, homotopic, combinatorial, and so on, which we have developed, we select the first. In the case where M is compact and F continuous a point p is termed *homotopically ordinary* if there exists a continuous deformation of some neighborhood of p which displaces p, and under which F is decreased whenever a point is displaced. A point not homotopically ordinary is termed critical. In the case where M is not compact and F merely semi-continuous we make the necessary technical modifications. Variational problems lead to spaces M in which the "point" is a curve, and the homotopic point is then termed a homotopic extremal. The fundamental theorem here is as follows. When the metric M is derived from the study of ordinary variational problems with a positive integrand on a compact differentiable space of class C^4 each homotopic extremal is an ordinary extremal. This theorem includes the Hilbert theorem that the absolute minimizing curve is an extremal, and it assumes nothing concerning the differentiability of the homotopic extremal. (Received December 1, 1936.)

47. Professor Rufus Oldenburger: *Real canonical binary trilinear forms.*

In earlier papers Dedekind, Gilham, others, and the author treated the problem of equivalence of binary trilinear forms under the class of non-singular linear transformations in the complex field. In the present paper there are obtained the canonical forms to which binary trilinear forms are equivalent under non-singular linear transformations in the field of reals and a complete invariant system. Five canonical forms are obtained, one more than for the complex field. The reductions to four canonical forms are given, while it is proved by the existence of roots of certain inequalities that forms not equivalent to these four are equivalent to the fifth. The complete invariant system obtained is of a rather simple character, comprising ranks defined in terms of 3-way matrices and properties of associated invariant factors. (Received November 27, 1936.)

48. Professor E. G. Olds: *Distributions of sums of squares of rank differences for small numbers of individuals.*

In a recent article (Annals of Mathematical Statistics, vol. 7 (1936), pp. 29–43) Hotelling and Pabst have indicated the need for the exact distribution of the coefficient of rank correlation for small values of n. From the article we have $i=1-6\sum d^2/(n^3-n)$, where i is the rank correlation coefficient, n the number of individuals, and $\sum d^2=\sum_{i=1}^n d_i^2$ (d_i being the rank difference for the ith individual). For n fixed in value, the distribution of i depends on the distribution of $\sum d^2$. This paper will be concerned with (1) the exact distribution of $\sum d^2$ for certain small values of n, (2) approximations obtained by using ordinates of the Pearson type II curve, (3) approximations obtained by using the normal curve, (4) the construction of a table to test the significance of rank correlation in small samples. (Received November 18, 1936.)

49. Professor Oystein Ore: *Extensions of the theorem of Jordan-Hölder in groups.*

It is shown how the theorem of Jordan-Hölder and the refinement theorem of Schreier-Zassenhaus may be extended to certain non-normal chains of subgroups. First a weak theorem is proved for "permutable" chains and next a stronger theorem by means of a new concept of quasi-normal subgroups. (Received November 14, 1936.)

50. Dr. J. F. Randolph: *Metric separability and the Hildebrandt integral.*

Two point sets are "metrically separated" if the outer measure of their sum is the sum of their outer measures (when each has outer measure finite). A function is "metrically separable" on a set A if for every constant the part of A where the function is greater than a constant is metrically separated from the rest of A. All measurable functions are metrically separable, but not conversely. It has been shown by Jeffery (Annals of Mathematics, vol. 33, pp. 443–459) that metric separability may be used as a basis for a comprehensive theory of integration (as suggested by Hildebrandt, this Bulletin, vol. 24 (1917), p. 128) which includes as special cases Young, Pierpont, and Lebesgue integration. Similar results for integration over plane sets with respect to Carathéodory outer linear measure do not follow from Jeffery's methods. The present paper develops new methods which envelop these broader applications. (Received December 1, 1936.)

51. Professor J. H. Roberts: *A metrization theorem.*

In the present paper it is proved that if S is a metric space and G is an upper semi-continuous collection of closed sets filling S (R. L. Moore's topological definition) then the space determined by the elements of G with the usual definition of limit element is metrizable. Furthermore if S is complete so is G. These results are obtained as consequences of the following new metrization theorem: Suppose $L_1(x, y)$ and $L_2(x, y)$ are functions defined for every x, y in a topological space E, and $L_i(x, y)$, $(i=1, 2)$, satisfies all the postulates for a distance function except the triangle inequality. Suppose furthermore that for any x, y, and z in E, $L_1(x, y)+L_1(y, z)\geqq L_2(x, z)$. Then E is metrizable. (Received November 23, 1936.)

52. Dr. S. L. Robinson: *Pseudo-k-fold transitive groups.*

A permutation group of degree n is pseudo-k-fold transitive if an unordered set of k letters can be changed into every other such unordered set by permutations of the group. In case n is large compared with k, it is shown that a pseudo-k-fold transitive group is $k-1$ times transitive. (Received December 1, 1936.)

53. Dr. A. R. Schweitzer: *Definitions of betweenness in the foundations of geometry.*

Dr. Schweitzer constructs two categories of definitions of betweenness in descriptive geometry (1) with reference to orientation and (2) primarily without reference to orientation. In the first category the author has given (American Journal of Mathematics, 1909) definitions of betweenness in terms of his relations R and K. Definitions of betweenness in the second category and associated descriptive systems of axioms are given in the present paper. These definitions are phrased for n equal to 1, 2, 3, . . . , in terms of the relations S_n, B_n, and I_n (loc. cit., p. 366). Definitions in terms of the relations S_n and B_n are constructed in analogy with those given by the author in terms of the relations R_n and K_n. Betweenness in terms of the relation I_n is defined in two ways, first in analogy with definitions of the oriented type and second in terms of the (defined) concept of a point lying on the boundary of an n-dimensional $n+1$ point. In the case of the relation I_n transition to metrical geometry is made by assuming that $n+1$ independent points are inscribed in a circle, sphere, hypersphere, . . . , the latter concepts being undefined under the axioms (see the reference to cyclic and spherical order, loc. cit., pp. 385, 389). Metrical definitions of betweenness have been given by Pieri, Huntington, and others. (Received November 25, 1936.)

54. Professor J. A. Shohat: *Application of Laguerre polynomials to Laplace integrals.*

Laplace integrals and Lorch's theorem are discussed by means of Parseval's formula for Laguerre polynomials. (Received December 1, 1936.)

55. Dr. Alvin Sugar: *A note on James' asymptotic Waring theorem for the cubic polynomial.*

By employing a transformation, the congruential condition of James' theorem is made less restrictive. (Received November 27, 1936.)

56. Professor Otto Szász: *On the absolute convergence of Fourier series.*

In 1934 S. Bernstein proved the theorem: Let $f(\theta)$ be a continuous periodic function of period 2π. Let c_n be the Fourier constants of $f(\theta)$. Define $\xi_f(\delta)$ by $\xi_f(\delta)=\max_{|\theta|\leqq\pi, |t|\leqq\delta}|f(\theta+t)-f(\theta)|$, $\delta>0$; then, if $\sum_{n=1}^{\infty} n^{-1/2}\xi_f(1/n)<\infty$, it follows that $\sum_{n=1}^{\infty}|c_n|<\infty$. Conversely, given a function $\eta(\delta)$ for which $\eta(\delta)/\delta^{\alpha}$ increases for sufficiently small α's and decreases for a certain $\alpha<1$, as δ increases; if, besides, the series $\sum n^{-1/2}\eta(1/n)$ diverges, then there exists a function $f(\theta)$ such that $\xi_f(\delta)<\eta(\delta)$ and $\sum|c_n|$ diverges. In this paper this result is generalized as follows: Let $f(\theta)$ be a function in L^p where $1<p\leqq 2$ with the period 2π. Define $\omega_f(\delta)$ by $\omega_f(\delta)=\max_{|t|\leqq\delta}[(1/2\pi)\int_{-\pi}^{\pi}|f(\theta+t)-f(\theta)|^p d\theta]^{1/p}$. Then, if $\sum n^{-(p-1)/p}\omega_f(1/n)<\infty$, it follows that $\sum|c_n|<\infty$. Conversely, given a function $\eta(\delta)$ of a certain type for which $\sum n^{-(p-1/)p}\eta(1/n)$ diverges, then there exists a function $f(\theta)$ for which $\omega_f(\delta)<\eta(\delta)$ and such that $\sum|c_n|$ diverges. The case $p=2$ includes Bernstein's theorem. In the proof of the first part, the inequalities of Hausdorff and Hölder are used. For the proof of the second part certain relations between the degree of approximation and the "modulus" of continuity in L^p arise. (Received November 23, 1936.)

57. Professor J. M. Thomas: *Functional dependence of polynomials.*

If $f_1, \cdots, f_{k+1}$ belong to the commutative polynomial ring $R[x_1, \cdots, x_n]$ and the rank of their functional matrix is k, a method is given for constructing in $R[f_1, \cdots, f_{k+1}]$ a non-zero polynomial $F(f_1, \cdots, f_{k+1})$, which vanishes as a member of $R[x_1, \cdots, x_n]$. Thus the function whose existence is stated by the dependence theorem of analysis can be taken as a polynomial when the given functions are all polynomials. (Received December 1, 1936.)

58. Professor H. S. Thurston: *On the irregular case of the quadratic equation in quaternions and binary matrices.*

Sylvester (Comptes Rendus, vol. 99, pp. 555–558, 621–631) showed that the equation $X^2+PX+Q=0$, where X, P, Q, are quaternions (or binary matrices), has six solutions, except when a certain cubic equation in a scalar variable, λ, has a root $\lambda=0$. This case he termed the *irregular* case. In this paper, a necessary and sufficient condition for the occurrence of the irregular case is obtained. It is found that the irregular case occurs if P and Q are linear functions of a quaternion (or binary matrix) A; in the quaternion equation this condition is also necessary. The paper discusses the number and nature of the solutions when the condition is satisfied. (Received December 1, 1936.)

59. Mr. R. E. Traber: *On simple convex neighborhoods.*

This paper presents a simpler proof of the existence of simpler convex neighborhoods in the geometry of paths, a theorem first proved by J. H. C. Whitehead (Quarterly Journal of Mathematics, vol. 3 (1932), pp. 33–42; vol. 4 (1933), pp. 226–227). It is based on a theorem on the domain of normal coordinates with origin at a point in a compact set, and uses the well known point-direction existence theorem on ordinary differential equations instead of the more involved two point form used by Whitehead. (Received November 27, 1936.)

60. Professor W. J. Trjitzinsky: *Analytic theory of non-linear singular differential equations.*

In this memoir, which will appear in the Mémorial des Sciences Mathématiques, the author develops an analytic theory of the non-linear differential equation of order n: $x^p y^{(n)}(x) = b(x, y, y^{(1)}, \cdots, y^{(n-1)})$, where p is a positive integer, and where the second member is analytic in the displayed arguments at $(x=0,\ y=y^{(1)}= \cdots = y^{(n-1)}=0)$. When b is linear in $y, y^{(1)}, \cdots, y^{(n-1)}$, we have a linear problem with a singularity of finite rank at $x=0$. The general analytic theory of equations of the latter type has been previously developed by the author of the present paper (Acta Mathematica, vol. 62, pp. 167–226). In the present work properties of solutions of the differential equation stated above are investigated in the complex neighborhood of the singular point $x=0$. (Received November 30, 1936.)

61. Professor A. W. Tucker: *Tensor algebra in topology.*

The algebra of the mappings and products of abstract complexes can be formulated advantageously in tensor terms. Correspondence numbers, or alternatively the coefficients of product chains, provide the tensor components. The boundary operator yields a sort of absolute differentiation. The relations between a complex and its dual reflect themselves of course in covariance and contravariance. The heart of the algebra is a non-commutative (essentially skew-commutative) contraction process which generalizes the notion of Kronecker intersection; inherent in this are a variety of products. (Received December 1, 1936.)

62. Dr. H. E. Vaughan: *On locally bicompact spaces.*

Two classes of locally bicompact spaces are defined each of which is a generalization of the class of locally compact separable metric spaces in the sense that each contains every space of the latter class and contains no other metric space although, in general, the spaces defined are not separable. Two additional characterizations of each class are obtained, one in terms of the covering theorems which hold in the spaces in question, the other in terms of the complements of the spaces with respect to the bicompact spaces in which they can be imbedded. It is shown that the spaces of one class are normal, while those of the other class are completely normal, and that many of the properties usually derived from perfect separability belong to the spaces of the latter class. (Received December 1, 1936.)

63. Professor J. L. Walsh: *On the shape of level curves of Green's function.*

If the point O lies interior to the Jordan curve C, we define the *circularity* of C as the quotient $K(C)$ of the shortest distance from O to a point of C by the longest distance from O to a point of C. Under a smooth conformal map $z=\psi(w)$ of the interior of the unit circle $|z|<1$ onto the interior of a Jordan curve in the w-plane, the circularity of C_r: $|\psi(w)| = r \leqq 1$ is non-decreasing as r decreases, approaches unity as r approaches zero, and satisfies the inequality $K(C_r) \geqq [K(C_1)]^{(4/\pi)\tan^{-1} r}$. (Received November 25, 1936.)

64. Professor Norbert Wiener: *A class of Tauberian theorems.*

Professor Wiener uses the method of undetermined functions to reduce a class of Tauberian theorems dealing with power series of rapid growth to the form where the kernel is a function of the difference of two variables. The functional equations thus derived are easily solvable. Among the results are certain gap theorems concerning power series of very rapid growth. It turns out that if this growth is sufficiently smooth the gap for the nth term cannot be of much greater order than the square root of n. (Received November 18, 1936.)

65. Professor F. L. Wren: *A theorem on determinant expansion and some applications.* Preliminary report.

A theorem on the expansion of determinants, which is a generalization of a theorem by Sylvester and also one by Chió, is established. This theorem is then applied to the solution of systems of linear equations, determining the rank of matrices, and the calculation of partial and multiple coefficients of correlation. (Received November 30, 1936.)

66. Professor Oscar Zariski: *Generalized weight properties of the resultant of $n+1$ polynomials in n indeterminates.*

Let $f_1, \cdots, f_{n+1}$ be $n+1$ polynomials in $x_1, \cdots, x_n$, of degrees $l_1, \cdots, l_{n+1}$ respectively, and let $s_1, \cdots, s_{n+1}$ be integers, $0 \leqq s_i < l_i$. The weight of any coefficient of f_i is defined as $s_i - j_i$, or zero, according as $j_i \leqq s_i$ or $j_i \geqq s_i$, where j_i is the degree of the corresponding term of f_i. It is proved that each term of the resultant $R = R(f_1, \cdots, f_{n+1})$ is of weight $\geqq s_1 s_2 \cdots s_{n+1}$. Moreover, if ϕ_i and ψ_i denote the sum of terms of f_i of degree $\leqq s_i$ and of degree $\geqq s_i$ respectively, then the sum of terms in R which are of weight $s_1 s_2 \cdots s_{n+1}$ is given by the expression $D^\sigma R(\phi_1, \cdots, \phi_{n+1})$, where D is the base of the (principal) ideal of the inertia forms of $\psi_1, \cdots, \psi_{n+1}$. Here $\sigma = 1$, except when all $s_i = l_i$ but one, say s_1, in which case $\sigma = l_1 - s_1$. These properties of the resultant lead to a direct algebraic derivation of the intersection multiplicity λ of $n+1$ hypersurfaces F_i in S_{n+1} at a common point P, of multiplicity s_i for F_i: $\lambda \geqq s_1 s_2 \cdots s_{n+1}$ and $\lambda = s_1 s_2 \cdots s_{n+1}$ if and only if the hypersurfaces have no common tangent at P. (Received December 1, 1936.)

67. Professor Oscar Zariski: *On the behavior of zero-dimensional ideals under quadratic transformations.*

A singular point P of an algebraic variety V in a linear space $S_n(x_1, x_2, \cdots, x_n)$ can be described by means of a suitable primary 0-dimensional ideal $\mathfrak{A}$ in the polynomial ring $K[x_1, \cdots, x_n]$, where K is the underlying field of constants. To resolve the singularity at P (assuming P to be the origin) it is customary to employ the quadratic transformation: $x_1' = x_1$, $x_i' = x_i/x_1$, $i = 2, \cdots, n$. That the singularity is actually simplified by the transformation is well known in the case of algebraic curves and in the case of surfaces in S_3—

and in these cases only. Proceeding in a purely arithmetic fashion, the author considers the length of the ideal $\mathfrak{A}$ as a controlling character of the singularity. Given an arbitrary primary 0-dimensional ideal $\mathfrak{A}$ in $K[x_1, \cdots, x_n]$, not necessarily defined by a singular point, a comparison is made of the length of $\mathfrak{A}$ with the length of each of the primary components $\mathfrak{A}_i'$ of the extended ideal of $\mathfrak{A}$ in the larger polynomial ring $K[x_1', \cdots, x_n']$. It is found that the length of each $\mathfrak{A}_i'$ is definitely less than the length of $\mathfrak{A}$ only if $n \leqq 3$, but that if $n > 3$, the length may actually be increased by the transformation. The positive part of the result furnishes at any rate an arithmetic proof for the resolution of the point singularities of algebraic surfaces. In view of the negative result in the case $n > 3$, it appears that a similar proof for higher varieties must take into consideration the special character of the ideals defined by point singularities. (Received December 1, 1936.)

68. Professor Oscar Zariski: *The topological discriminant group of a Riemann surface of genus p and an application to the Poincaré group of plane elliptic curves.*

By the *topological discriminant group of degree n* of a complex K is meant the Poincaré group of $K^n - D$, where K^n is the symmetric nth product of K and D is the subcomplex of K^n representing the n-tuples with two or more coincident points. This group, G_n, is explicitly determined when K is a Riemann surface of genus p (for $p = 0$, see the author's paper in the American Journal of Mathematics, 1936). In the case $p = 1$ it is shown that the commutator group of G_n gives the Poincaré group of the maximal cuspidal plane elliptic curve of order $2n$. This result is applied toward the determination of the Poincaré group of an arbitrary plane elliptic curve with cusps and nodes. The special case $n = 3$ gives rise to a topological interpretation of the properties of the configuration of the flexes of a plane elliptic cubic. (Received December 1, 1936.)

69. Dr. J. A. Greenwood: *Restricted absolute permutations.*

The problem is equivalent to finding the number of non-vanishing terms and value of a determinant of ones and zeros of order n having n zeros down the principal diagonal and s other zeros subject only to the restriction that every principal first minor have $\geqq n+s-2$ zeros. It is found that the number of terms $N_n(s) = \sum_{i=1}^{s} (-1)^s C_{s,i} N_{n-i}^{n-2i}$, where the numbers $N_m^q = \sum_{r=0}^{q} (-1)^r C_{q,r}\ (m-r)!$ were evaluated by Longchamps (1891). The value of the determinant is found to be $(-1)^n \cdot (1+s-n)$. From these two results the exact number of even and odd permutations of this restricted absolute set is easily obtained. This is a generalization of the case $s = 0$ published by J. M. Thomas in this Bulletin, 1925. The result, $\lim_{n\to\infty} N_n(0)/n! = \exp\ (-1)$, which is due to Cantor, is now generalized to $[\exp\ (-1) \geqq \exp\ (-1 - \underline{\phi(n)/n}) = N_n[\phi(n)]/n! \geqq \exp\ (-1 - \overline{\phi(n)/n}) \geqq \exp\ (-3/2)]_{(n\to\infty)}$. (Received November 27, 1936.)

70. Mr. I. Halperin and Professor John von Neumann: *On the transitivity of perspective mappings in complemented modular lattices.*

Transitivity is an essential property of the equivalence by perspectivity in continuous geometries (see J. von Neumann, Proceedings of the National

Academy, 1936, pp. 92–101). Its proof is rather complicated, and uses the transcendent "continuity axioms" of this theory. The general transitivity is established by proving it first for non-intersecting elements. In the present note a simpler proof of this last fact is given, a great part of the proof being quite elementary and independent of the "continuity axioms," hence valid for all complemented modular lattices. (Received December 2, 1936.)

71. Mr. I. Halperin: *On the transitivity of perspectivities in continuous geometries.*

The transitivity of perspectivity (see abstract 43-1-70) is established by a simpler method, and without using the irreducibility of the continuous geometry. The additivity and continuity properties of equivalence by perspectivity are also obtained more briefly. (Received December 2, 1936.)

72. Dr. Olaf Helmer: *Polynomials in infinitely many variables.*

Let $x_1, x_2, \cdots, x_n, \cdots$ be a sequence of infinitely many variables, running through all sequences $a_1, a_2, \cdots$ for which $\sum |a_n|$ converges. $f(x_1, x_2, \cdots)$ is called a *polynomial* in $x_1, x_2, \cdots$ if, roughly, it is a polynomial in the ordinary sense in each of the variables x_1, x_2, and so on. Thus, $x_1+x_2^2+x_3^3+\cdots$ and $x_1+x_1x_2+x_1x_2x_3+\cdots$ are polynomials in this sense, but $x_1+x_1^2x_2+x_1^3x_3+\cdots$ is not. The symbols 1, 2, 3, $\cdots$, taken in any order of the ordinal type ω, are said to form a *permutation* of 1, 2, 3, $\cdots$. Let the notation be similar to the customary one: $P=\binom{1\ 2\ 3\ \cdots}{a_1a_2a_3\ \cdots}$. The function resulting from the application of P to the arguments of $f(x_1, x_2, \cdots)$ may be designated by $f_p(x_1, x_2, \cdots)$. The latter is a polynomial if, and only if, the former is one. f is called *symmetrical*, if $f_p=f$ for every P. The *elementary symmetrical functions* can now be defined as usual; they are polynomials in our sense. The *main theorem on symmetric functions* can be extended to polynomials as defined here. Most of the usual proofs fail, as they involve complete induction with respect to the number of variables. The Gauss-Waring proof, however, can be so modified as to fit the requirements. (Received December 1, 1936.)

73. Dr. Nathan Jacobson (National Research Fellow): *Abstract derivation and Lie algebras.*

Let $\mathfrak{A}$ be an arbitrary algebra (not necessarily associative or commutative or of finite order) over a commutative field. We define a *derivation D* in $\mathfrak{A}$ as a linear mapping of $\mathfrak{A}$ into itself such that $(xy)D=(xD)y+x(yD)$. The set of derivations is shown to be a Lie algebra $\mathfrak{D}$ (with a slight restriction when $\mathfrak{A}$ has characteristic $\neq 0$). In the first part of this paper we discuss some elementary properties of $\mathfrak{D}$. Next we determine $\mathfrak{D}$ for a semi-simple associative algebra with a finite basis. A fundamental result is that if $\mathfrak{A}_1$ is a semi-simple subalgebra of a normal simple algebra, then any derivation in $\mathfrak{A}_1$ may be extended to an *inner* derivation in $\mathfrak{A}$, that is, a derivation of the form $x\rightarrow xd-dx\equiv xD$. Some of the algebras obtained in the case of characteristic $\neq 0$ are semi-simple Lie algebras which are not direct sums of simple algebras. If $\mathfrak{A}$ is an inseparable field, $\mathfrak{D}$ is a simple algebra of a type which has no counterpart for characteristic 0. We discuss this algebra and its relation to $\mathfrak{A}$ in greater detail, obtaining a number of results analogous to well known theorems on automorphisms of fields. For example, we give a characterization of the logarithmic derivatives of $\mathfrak{A}$ which is analogous to Hilbert's theorem on the elements of norm 1 in a cyclic field. (Received December 1, 1936.)

74. Dr. Nathan Jacobson (National Research Fellow): *A class of normal simple Lie algebras of characteristic* 0.

Let $\mathfrak{A}$ be an involutorial normal simple algebra over a field Φ of characteristic 0 and J an involutorial anti-automorphism in $\mathfrak{A}$. The set $\mathfrak{S}_J$ of *J-skew* elements $a(a^J=-a)$ is closed under addition, scalar multiplication, and commutation. We show that $\mathfrak{S}_J$ but for one trivial exception is a simple Lie algebra which becomes either the complex or orthogonal Lie algebras when Φ is extended to its algebraic closure Ω. $\mathfrak{S}_{J_1}$ and $\mathfrak{S}_{J_2}$ are isomorphic if and only if the associative algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ defining them are isomorphic and J_1 and J_2 are *cogredient* in the sense that $J_1=S^{-1}J_2S$ where S is an isomorphism of $\mathfrak{A}_2$ into $\mathfrak{A}_1$. The automorphisms of $\mathfrak{S}_J$ are realized as transformations $a\rightarrow g^{-1}ag$ where g is *J-orthogonal* ($g^Jg=\gamma_1\neq 0$ in Φ). Any Lie algebra $\mathfrak{L}$ over Φ such that $\mathfrak{L}_\Omega$ is isomorphic to the complex or orthogonal Lie algebras may be realized as an algebra $\mathfrak{S}_J$. The question of cogredience of anti-automorphisms is essentially one of ordinary cogredience of hermitean and skew-hermitean matrices with elements in a normal division algebra. We hope to consider this problem in a later paper. (Received December 1, 1936.)

75. Professor E. J. McShane: *A navigation problem in the calculus of variations.*

An airplane is to travel from one point to another in the least possible time. Its velocity relative to the air is required to lie in a convex set $K(x, t)$ of vectors depending on the place x and the time t. The air velocity is a continuous vector function $u(x, t)$. Existence theorems are established for this problem and for the modified form of the problem in which the velocity is required to be almost always as great as possible for the time, place, and direction of the airplane. This is a generalization of the Zermelo navigation problem. (Received December 2, 1936.)

76. Professor E. J. McShane: *Jensen's inequality.*

Jensen's inequality is established in the following general form. If L is a linear class of functions $f(x)$ and Mf is a linear mean-value functional on L such that $M1=1$ and $Mf\geqq 0$ if $f\geqq 0$, then every convex function $\phi(z_1, \cdots, z_n)$ satisfies the inequality $\phi(Mf_1, \cdots, Mf_n)\leqq M\phi(f_1, \cdots, f_n)$. Conditions for equality are investigated. (Received December 2, 1936.)

77. Dr. M. S. Robertson: *A representation of all analytic functions in terms of functions with positive real part.*

Any analytic function by trivial transformations can be normalized to the form $f(z)=z^k+\sum_{k+1}^{\infty}a_nz^n$, $k\geqq 1$, holomorphic for $|z|<1$ with the property that for each $r<1$ the map of $|z|=r$ is a contour cut by the real axis in exactly $2k$ points. It is shown how such a function can be completely characterized in terms of a function with positive real part in the unit circle and $2k$ parameters θ_i whose values depend upon $f(z)$. From this representation one deduces that $|a_n| \leqq (n/k) \cdot (n+k-1)!/(2k-1)!(n-k)! = O(n^{2k})$ where the equality sign is attained by essentially only one function of this class. If $f(z)$ is also real on the real axis $a_n = O(n^{2k-1})$. If $g(z) = z + \sum_2^\infty c_n z^n$ is regular for $|z| < 1$ and star-like in $2k$ symmetric directions with respect to the origin (but not necessarily univalent) then $c_n = O(n^{1+(1/k)})$. For the functions $f(z)$ and $g(z)$ the radial limits $(r \to 1)$ exist finitely almost everywhere. (Received November 30, 1936.)

78. Dr. W. E. Sewell: *Degree of approximation by polynomials —problem α.*

Let C be a Jordan curve in the z-plane and let $z = \psi(w)$ map the exterior of C on $|w| > 1$, so that the points at ∞ correspond to each other. Let $\psi''(w)$ be continuous in $|w| \geqq 1$. *Let $f(z)$ be analytic in C, and satisfy a Lipschitz condition of order α, $0 < \alpha \leqq 1$, in $\overline{C}$, the closed limited point set bounded by C. Then there exist polynomials $P_n(z)$, $n = 2, 3, \cdots$, of degree n in z, such that $|f(z) - P_n(z)| \leqq M$ (log $n/n)^\alpha$, z in $\overline{C}$, where M is a constant independent of n and z.* (Received December 1, 1936.)

79. Professor D. J. Struik: *On surface theory in four space.*

Different possibilities of generalizing the ordinary surface theory to four space are sketched, with special reference to developable surfaces and ruled surfaces in general. An application is given to certain surfaces of zero curvature. (Received December 1, 1936.)

80. Professor G. T. Whyburn: *On interior transformations.*

Let $T(A) = B$ be an interior transformation, where A is compact and metric. Then (i) for any subset Q of A satisfying $Q = T^{-1}T(Q)$, T is an interior transformation on Q; (ii) for any continuum $C \subset B$, every component of $T^{-1}(C)$ transforms into the whole of C; (iii) for any open set R in A, $F[T(R)] \subset T[F(R)]$, where $F(X)$ denotes the boundary of X; (iv) if A is locally connected, B_0 is any closed set in B, and Q is any component of $B - B_0$, then $T^{-1}(Q)$ is the sum of a finite number of components of $A - T^{-1}(B_0)$ each of which maps into exactly Q. These results yield a complete analysis of interior transformations defined on a number of particular types of sets A. For example, if A is a simple closed curve, B must be either a simple closed curve or an arc; and in the former case T is completely alternating and hence is equivalent to the transformation $w = z^k$ on $|z| = 1$, while in the latter, T is equivalent to the transformation $f(x) = \cos k\pi x$ of $(0, 1)$ into $(-1, 1)$. (Received January 5, 1937.)

81. Dr. J. H. Chanler: *The involution curve determined from a special pencil of n-ics.*

The author investigates the involution curve C_k determined by k-ads from a pencil of binary n-ics which contains one member with j double roots. The study was suggested by the curves $[W_p, S_p]$ of W_p (A. B. Coble and J. H. Chanler, *The geometry of the Weddle manifold W_p*, American Journal of Mathematics, vol. 57 (1935), pp. 183–218). The genus and multiple points of C_k are determined by applying Zeuthen's formula to the correspondence set up between the points of C_k and the points of the norm curve. (The method of counting branch points used in §§2, 6 is due to Professor O. Zariski.) The curve and manifolds associated with it are also studied from the projective viewpoint. By this means manifolds investigated by Brill are proved to be special cases of those studied by Salmon, Cayley, and others. Lastly cases are discussed in which some of the j double roots of the special n-ic coincide. (Received December 7, 1936.)

82. Professor N. A. Court: *Desmic tetrahedrons related to quadric surfaces.*

Given the tetrahedron (T) inscribed in the sphere (S), A. A. Bennett determined analytically the tetrahedron (K) self-polar both with respect to (S) and (T) (American Mathematical Monthly, 1932, pp. 18–27). A synthetic discussion of this problem, in which the sphere (S) is replaced by a non-ruled quadric (Q), leads to the following construction of (K). Let a, a' be a pair of opposite edges of (T), x, x' their polar lines with respect to (Q), and u, u' the two lines meeting the lines a, a', x, x'. The lines u, u' are unique and are always real. If K, K' are the double elements of the involution determined by the two pairs of points ua, ua'; ux, ux', and K'', K''' the two analogous points on u', the points K, K', K'', K''' are the vertices of (K). The tetrahedron (T'), which with (T) and (K) forms a desmic system, is inscribed in (Q) and is the only tetrahedron inscribed in (Q) and self-polar with respect to (T). (Received December 3, 1936.)

83. Dr. S. C. Kleene: *On notation for ordinal numbers.* Preliminary report.

There exists a *greatest* ordinal ξ in the second number class such that there is a system of notation in which (1) at least one formula X is assigned to represent each ordinal $x<\xi$, (2) no X represents two distinct ordinals, (3) from X it is effectively decidable whether $x=0$ or $x=y+1$ or $x=\lim_\nu\{x_\nu\}$, (4) if $x=y+1$, then from the formula X a formula Y representing y is effectively calculable, (5) if $x=\lim_\nu\{x_\nu\}$, then from X a sequence of formulas $x_1, x_2, \cdots$ representing $x_1, x_2, \cdots$, respectively, is effectively calculable. (Received December 29, 1936.)

84. Professor R. G. Lubben: *Concerning perfectly normal Hausdorff spaces.*

1. In order that a Hausdorff space should be perfectly normal, it is necessary and sufficient that each closed set in it be both the product and the limiting set of a countable collection of open sets. 2. In order that a Hausdorff space having the Lindelöf property should be perfectly normal it is necessary and sufficient that it be regular. 3. In a Hausdorff space having the Lindelöf property, in order that every point set should have the Lindelöf property it is sufficient that the space be perfectly normal. (Received December 29, 1936.)

85. Professor H. A. Rademacher: *On the partition function.*

In 1917 Hardy and Ramanujan gave an asymptotic formula for the number $p(n)$ of unrestricted partitions of n. A revision and extension of their method leads to a convergent series for $p(n)$, of which the asymptotic formula is an immediate consequence. (Received December 21, 1936.)

86. Mr. L. B. Robinson: *A system of Riquier and the tensor calculus. II.*

The complete system (1)$\sum_{j=1}^{r} P_{ij}(x_1 x_2 \cdots x_j)\partial f/\partial x_{j+1}=0$, $(i=1, 2, \cdots, n)$, where the P are polynomials, shall be called a system of class K. Solve the final equation of the system by quadratures. The solutions $w_2 \equiv x_1, w_3, w_4, \cdots, w_{r+1}$ can serve as new variables and can be substituted in the remaining equations, $n-1$ in number. The author now has a new system $\sum_{j=2}^{r} Q_{ij}(w_2, w_3, \cdots, w_j) \cdot \partial f/\partial w_{j+1}=0$ which is also of class K. Continue and solve the entire system by quadratures. The final solution is written: $v_1 \equiv x_1$, $v_2 \equiv x_{n+2}+f_{n+2}(x_1, x_2, \cdots, x_n, x_{n+1})$, $v_3 \equiv x_{n+3}+f_{n+3}(x_1, x_2, \cdots, x_n, x_{n+1}, x_{n+2}), \cdots, v_{r+1} \equiv x_{r+1-n}+f_{r+1}(x_1, x_2, \cdots, x_r)$ where the f are rational with respect to x_1 and rational and integral with respect to the other variables. In next resumé the author plans to extend these results to a more general class and then apply the theorems to the tensor calculus. (Received December 29, 1936.)

87. Professor D. V. Widder: *The Stieltjes transform.*

This paper is concerned with the transform $f(x)=\int_0^\infty (x+t)^{-1}d\alpha(t)$. A real inversion formula analogous to the Taylor determination of the coefficients of a power series is obtained. We define $L_{k,t}[f(x)]$ as a linear differential operator whose fundamental solutions are $x^n(n=-k, -k+1, \cdots, -1, 0, 1, \cdots, k-2)$. If $\alpha(t)$ is the integral of a function $\phi(t)$, then $L_{k,t}[f(x)]$ approaches $\phi(t)$ for almost all positive values of t. If $\alpha(t)$ is a normalized function of bounded variation, the integral of this same operator between the limits 0 and t approaches $\alpha(t)-\alpha(0+)$ for all positive t. Necessary and sufficient conditions on $f(x)$ that it should be the transform of a function $\alpha(t)$ are obtained. Finally the relation of the above inversion formula to one given recently by Paley and Wiener is investigated. Symbolically, the latter is $(\pi)^{-1}t^{-1/2}(\cos \pi D)(t^{1/2}f(t))$, where D is the operation of differentiation with respect to t. Paley and Wiener use the power series development of the cosine, whereas $L_{k,t}[f]$ is a section of the familiar infinite product development of the cosine. (Received December 15, 1936.)

88. Professor E. L. Dodd: *Internal and external means arising from the scaling of frequency functions.*

It is well known that a scale obtained in a curve-fitting process is sometimes a mean. For example, with the normal function, constant $\cdot \exp\{-(x/a)^2/2\}$, the scale a is usually taken as the root-mean-square of the given measurements. The R. A. Fisher maximum likelihood formula, indeed, leads to just this scale. In scaling by the Fisher method, however, the usual criterion sometimes leads to minimum likelihood instead of to maximum likelihood. And scaling is not always unique—sometimes a scale with minimum likelihood will be an internal mean; and a scale with maximum likelihood, an external mean. Here a generalized mean $F(x_1, x_2, \cdots, x_n)$ is used, subject to the restriction that if c is a value which the x's can take on, then $F(c, c, \cdots, c)=c$. Using the likelihood method, conditions are given (1) that a scale known as existing will be a *mean*; (2) that such a scale will be an *internal* mean as regards absolute values; and (3) that a scale will *exist*. The theory is illustrated largely from Pearson types. (Received January 4, 1937.)

89. Dr. J. M. Feld: *Plane cubic curves invariant under quadratic transformations.*

In a paper by K. Ogura, (Tôhoku Mathematical Journal, 4 (1913), p. 132), it was shown that the non-singular plane cubics invariant under the quadratic transformation $x_1':x_2':x_3'=x_2x_3:x_3x_1:x_1x_2$, fall into two classes bearing the equations A: $\sum a_1x_1(x_2^2-x_3^2)=0$ and B: $\sum a_1x_1(x_2^2+x_3^2)+a_4x_1x_2x_3=0$. It is proved in this report that every non-singular cubic is invariant under this quadratic transformation and, moreover, that every non-singular cubic may bear equation A or B, according to the manner in which the fundamental triangle is selected. (Received December 31, 1936.)

90. Dr. Aaron Fialkow (National Research Fellow): *Systems of plane curves.*

The author studies systems of ∞^2 curves which have the Cesàro-Scheffers property that the centers of the ∞^1 circles which osculate those curves passing through a common point lie on a straight line. These two-parameter systems include the isogonal trajectories of any one-parameter family and the extremals of a common calculus of variations problem as two important types. A consideration of the envelope of any ∞^1 Cesàro-Scheffers lines leads to a projectivity. It is characteristic of the extremals that this projectivity is always an involution. This result also gives a new characterization of isothermal curves. Isogonal systems may be completely described by means of Lamé's relation. If the two-parameter family is isogonal, a simple geometric construction of the projectivity is given. In this case, the hyperosculating circles play an important role. A separate discussion is given when the totality of Cesàro-Scheffers lines degenerates into a one-parameter family. These results are applied to characterize one-parameter families of curves, $u(x, y)=c$, where $u(x, y)$ is a solution of a given partial differential equation. (Received December 17, 1936.)

91. Professor H. A. Rademacher: *A convergent series for the partition function $p(n)$.*

The method by which in 1917 Hardy and Ramanujan obtained an asymptotic formula for $p(n)$ can be modified in such a way as to give an exact expression for $p(n)$ in the form of an infinite series. This determination for $p(n)$ in turn leads to a new representation for the generating function $f(x)=\sum_0^\infty p(n)x^n$ in which its natural boundary is dissolved into distinct singularities. (Received January 7, 1937.)

92. Dr. A. R. Schweitzer: *A theory of oriented angles in the foundations of geometry.*

In verification of a conjecture made by the author in this Bulletin, vol. 15 (1908), p. 81, a system of axioms is constructed for descriptive geometry which is equivalent to the descriptive theory of the author in the American Journal of Mathematics, vol. 31 (1909), pp. 365–411. The present system is phrased in terms of an undefined relation K expressing concretely sameness of sense of two n-dimensional angles, α_{jn} and β_{jn}, where $1 \leqq j < n(n=2, 3, \cdots)$ and $\alpha_{jn} = (\alpha_1\alpha_2 \cdots \alpha_{n+1-j}, \alpha_1\alpha_2 \cdots \alpha_{n+2-j}, \cdots, \alpha_1\alpha_2 \cdots \alpha_{n+1})$, and similarly for β_{jn}. This descriptive theory is made complete for euclidean and lobachevskian geometry. In a second part of the paper the preceding theory of oriented angles is restated in terms of the "quotient relation," $\alpha_{jn}/\beta_{jn} = k$, where k is a real number, and is then completed in analogy with the author's theory in the American Journal of Mathematics, vol. 35 (1913), pp. 37–56. This theory of oriented angles combined with the latter theory of oriented areas, volumes, and so on, provides an adequate theory of congruence in Grassmann's extensive algebra. Metrically, the theory of angles in the foundations of geometry has been studied by E. Study (1885), D. K. Picken (1924), C. Boccallotte (1929), G. Hessenberg (1930), and others. (Received January 8, 1937.)

93. Professor W. E. Roth: *On certain matrices and their determinants.*

It is here shown that the $nr \times nr$ matrix $M = \langle A_0 \rangle I + \langle A_1 \rangle U + \cdots + \langle A_{r-1} \rangle U^{r-1}$ where A_i, $(i=0, 1, \cdots, r-1)$, are $n \times n$ matrices, where U is an $r \times r$ matrix, and where $\langle A_i \rangle U^i$, $(i=0, 1, \cdots, r-1)$, is the direct product of A_i and U^i, is equivalent to the direct sum of the r matrices $A_0 + A_1\theta_j + \cdots + A_{r-1}\theta_j^{r-1}$, $(j=1, 2, \cdots, r)$, where θ_j are the roots of $|U - I\lambda| = 0$. As a result $|M|$ is the product of the r determinants $|A_0 + A_1\theta_j + \cdots + A_{r-1}\theta_j^{r-1}|$, $(j=1, 2, \cdots, r)$. If the A_i are scalars, we have a generalization of the well-known results given under cyclic determinants by Pascal in *Die Determinanten.* (Received December 30, 1936.)