Two rational automorphs A_1 and A_2 of f may be called *right-equivalent* if there exists an integral automorph I of f such that $A_1 I = A_2$. If (T_1, U_1) and (T_2, U_2) belong to the same set* of solutions of (5), the corresponding automorphs (with $t = T_i/m$, $u = U_i/m$) are readily seen to be right-equivalent.

McGill University

A NOTE ON RECURSIVE FUNCTIONS†

BY S. C. KLEENE

The notion of a recursive function of natural numbers, which is familiar in the special cases associated with primitive recursions, Ackermann-Péter multiple recursions, and others, has received a general formulation from Herbrand and Gödel. The resulting notion is of especial interest, since the intuitive notion of a "constructive" or "effectively calculable" function of natural numbers can be identified with it very satisfactorily.‡ Consider the operation of passing from a function $\rho(x_1, \cdots, x_n, y)$, such that for each set of values of $x_1, \cdots, x_n$ the equation $\rho(x_1, \cdots, x_n, y) = 0$ has solutions for y, to the function "$\epsilon y[\rho(x_1, \cdots, x_n, y) = 0]$" of which the least solution is $x_1, \cdots, x_n$. We have shown that the (general) recursive functions are the functions which are derivable from the primitive recursive functions by one application of this operation and of substitution.§ Herein we note the related result, that the recursive functions are the functions obtainable by repeated applications of the operation just described and of substitution from the three particular functions $x+y$ (sum), $x \cdot y$ (product), δ_y^x (Kronecker delta). This result follows from the other by an adaptation of an argument used by Gödel in proof that every

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* For definition of set, see Pall, Transactions of this Society, vol. 35 (1933), p. 491; or Dirichlet, *Vorlesungen über Zahlentheorie*, §87.

† Presented to the Society, January 1, 1936.

‡ See A. Church, *An unsolvable problem of elementary number theory*, American Journal of Mathematics, vol. 58 (1936), pp. 345–363, §7.

§ S. C. Kleene, *General recursive functions of natural numbers*, Mathematische Annalen, vol. 112 (1936), No. 5, IV and V.

primitive recursive relation is arithmetic, which we now present.* In doing so, we duplicate in part the analysis of primitive recursive functions given by Hilbert and Bernays.† For the notation and terminology, see our cited article.

Definition 1. Let C denote the least class of functions such that $x+y$, $x\cdot y$, δ_y^x, $U_i^n(x_1, \cdots, x_n)\,\epsilon C$, if $\theta(y_1, \cdots, y_m)$, $\chi_1(\mathfrak{x}), \cdots, \chi_m(\mathfrak{x})\epsilon C$, then $\theta(\chi_1(\mathfrak{x}), \cdots, \chi_m(\mathfrak{x}))\epsilon C$, and if $\rho(\mathfrak{x}, y)\epsilon C$ and $(\mathfrak{x})(Ey)[\rho(\mathfrak{x}, y)=0]$, then $\epsilon y\,[\rho(\mathfrak{x}, y)=0]\epsilon C$.

Instead of the identity functions $U_i^n(x_1, \cdots, x_n)=x_i$, a second substitution schema $\theta(\chi_1(\mathfrak{x}), \chi_2(\mathfrak{y}))$ may be introduced.

Using the identity functions, we can replace substitutions which do not fall under the given schema by series of substitutions which do thus, $\theta(x, \phi(x, y, z), \psi(z, y))=\theta(U_1^3(x, y, z), \phi(x, y, z), \psi(U_3^3(x, y, z), U_2^3(x, y, z)))$.

Definition 2. A relation ϵC if its representing function ϵC.

Analogously to Gödel's I–IV:

I. The functions $C(x)$, $S(x)$, $U_i^n(x_1, \cdots, x_n)\epsilon C$. Every function (relation) obtained by substitution of functions which ϵC for variables of functions (relations) which $\epsilon C \epsilon C$. If the relation $R(\mathfrak{x}, y)\epsilon C$ and $(\mathfrak{x})(Ey)R(\mathfrak{x}, y)$, then $\epsilon y[R(\mathfrak{x}, y)]\epsilon C$.

II. If the relations R, $S\epsilon C$, so do $\overline{R}$, $R\vee S$ (hence also $R\&S$).

III. If the functions $\phi(\mathfrak{x})$, $\psi(\mathfrak{y})\epsilon C$, so does the relation $\phi(\mathfrak{x})=\psi(\mathfrak{y})$.

IV. If the function $\phi(\mathfrak{x})$ and the relation $R(x, \mathfrak{y})\epsilon C$, so do the relations $(x)[x<\phi(\mathfrak{x})\rightarrow R(x, \mathfrak{y})]$ and $(Ex)[x<\phi(\mathfrak{x})\&R(x, \mathfrak{y})]$.

Proofs. $C(x)=\epsilon y[x\cdot y=0]$. Let $\alpha(x)=\epsilon y[\delta_y^x=0]$. Then $\alpha(0)=1$, $\alpha(1)=0$. $S(x)=x+\alpha(C(x))$. For II we see that, if $\rho(\mathfrak{x})$ and $\sigma(\mathfrak{y})$ are the representing functions of R and S, $\alpha(\rho(\mathfrak{x}))$ is the representing function of $\overline{R}$ and $\rho(\mathfrak{x})\cdot\sigma(\mathfrak{y})$ is the representing function of $R\vee S$. Let $\gamma(x, y)=\alpha(\delta_y^x)$. Then $\gamma(x, x)=0$ and $\gamma(x, y)=1$ for $x\neq y$. III holds since $\gamma(\phi(\mathfrak{x}), \psi(\mathfrak{y}))$ is the representing function of $\phi(\mathfrak{x})=\psi(\mathfrak{y})$. In proof of IV, we have

$$(x)[x<\phi(\mathfrak{x})\rightarrow R(x, \mathfrak{y})]\equiv \epsilon x[\overline{R(x, \mathfrak{y})}\vee x=\phi(\mathfrak{x})]=\phi(\mathfrak{x}),$$

$$(Ex)[x<\phi(\mathfrak{x})\&R(x, \mathfrak{y})]\equiv \overline{(x)}[x<\phi(\mathfrak{x})\rightarrow \overline{R(x, \mathfrak{y})}].$$

V. $x<y\epsilon C$. $x\equiv y(\mathrm{mod}\ n)\epsilon C$.

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* Kurt Gödel, *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I*, Monatshefte für Mathematik und Physik, vol. 38 (1931), pp. 173–198, Satz VII.

† Hilbert-Bernays, *Grundlagen der Mathematik*, vol. 1, pp. 412–421.

For

$$x<y\equiv\overline{(Ez)}\,[z<x+1\,\&\,x=y+z],\ [x\equiv y(\mathrm{mod}\ n)]$$
$$\equiv (Ez)\,[z<x+y+1\,\&\,\{x=y+z\cdot n\vee y=x+z\cdot n\}\,],$$

(see Gödel, p. 191).

VI. If the relation $x_0=\phi(\mathfrak{x})\epsilon C$, the function $\phi(\mathfrak{x})\epsilon C$.

For $\phi(\mathfrak{x})=\epsilon x_0[x_0=\phi(\mathfrak{x})]$.

VII. If $\phi(\mathfrak{x})$ is a primitive recursive function, the relation $x_0=\phi(\mathfrak{x})\epsilon C$.

The proof is by induction with respect to the number of applications of schemas (1) and (2) in the primitive recursive definition of $\phi(\mathfrak{x})$ (see Kleene, Def. 1). First, the proposition $x_0=\phi(\mathfrak{x})\epsilon C$ holds if $\phi(\mathfrak{x})$ is one of the functions $S(x)$, $C(x)$, $U_i^n(x_1, \cdots, x_n)$, by I and III. Second, if it holds for $\theta(y_1, \cdots, y_m)$, $\chi_1(\mathfrak{x}), \cdots, \chi_m(\mathfrak{x})$, then $\chi_1(\mathfrak{x}), \cdots, \chi_m(\mathfrak{x})\epsilon C$ by VI, and it holds for $\theta(\chi_1(\mathfrak{x}), \cdots, \chi_m(\mathfrak{x}))$ by I. Third, suppose Gödel, p. 192, 2 holds, where $\psi(x_2, \cdots, x_n)$, $\mu(x_1, \cdots, x_{n+1})$ have the property in question. Following Gödel's discussion, Hilfsatz 1 holds with $n>d$, since d is determined first, and n is any number which satisfies the congruences $n\equiv f_i[\mathrm{mod}(1+(i+1)d)]$, $(i=0, \cdots, k-1)$. Again, $x=[n]_p\epsilon C$. If $P(x_0, \cdots, x_n)\equiv x_0=\phi(x_1, \cdots, x_n)$, $S(x_0, x_2, \cdots, x_n)\equiv x_0=\psi(x_1, \cdots, x_n)$, $T(x_0, \cdots, x_{n+1})\equiv x_0=\mu(x_1, \cdots, x_{n+1})$, the relation on page 193 can be restated as follows: $P(x_0, \cdots, x_n)\equiv x_0=[N]_{1+D(x_1+1)}$, where N stands for $\epsilon n[(Ed)[d<n\,\&\,\{S([n]_{d+1}, x_2, \cdots, x_n)\,\&\,(k)[k<x_1\rightarrow T([n]_{1+d(k+2)}, k, [n]_{1+d(k+1)}, x_2, \cdots, x_n)]\}]]$ and D stands for $\epsilon d\{S([N]_{d+1}, x_2, \cdots, x_n)\,\&\,(k)[k<x_1\rightarrow T([N]_{1+d(k+2)}, k, [N]_{1+d(k+1)}, x_2, \cdots, x_n)]\}$. It follows that $P(x_0, \cdots, x_n)\epsilon C$.

VIII. Every primitive recursive function ϵC (by VI, VII).

IX. Every recursive function ϵC (by VIII, I, Kleene IV).

X. Every function which ϵC is recursive.

This follows from the recursiveness of $x+y$, $x\cdot y$, δ_y^x and functions obtained from recursive functions by substitution, and Kleene V. Combining IX and X, we have the following result.

XI. The class C of functions is identical with the class of recursive functions.

THE UNIVERSITY OF WISCONSIN