A GENERALIZATION OF WARING'S PROBLEM*

BY L. E. DICKSON

1. *Introduction.* Define $g(n, m)$ so that every integer $\geqq m$ is a sum of g nth powers, while not every integer $\geqq m$ is a sum of $g-1$ powers. It is customary to write $g(n)$ for $g(n, 0)=g(n, 1)$. Quite recently I evaluated $g(n)$ for every $n>6$.

For $n=9$ or 11, I evaluate $g(n, m)$ for each m below specified large values M. In particular, $g(11, M)=336$ and $g(9, M)=163$ are small compared with $g(11)=2132$ and $g(9)=548$.

By use of the Hardy-Littlewood Theory and extensive tables, it was found that $g(6) \leqq 160$. I here obtain $g(6) \leqq 110$.

2. *Asymptotic Theory.* Recently I proved† that, if $n \geqq 4$, every integer $\geqq N$ is a sum of $s-2+3k$ integral nth powers $\geqq 0$, where the quantities are defined as follows. Let p^θ be the highest power of the prime p which divides n. Write $\gamma=\theta+1$ if $p>2$, $\gamma=\theta+2$ if $p=2$. Let D be the g.c.d. of $p-1$ and n/p^θ. Write $m=D(p^\gamma-1)/(p-1)$. Then the conditions on s (for Lemma B of A) are $s>2n$, $s \geqq m+1$, for every prime p. They hold for $s \geqq 13$ if $n=6$, and for $s \geqq 2n+1$ if n is an odd prime or its square.

Employ natural logarithms. If $\vartheta(b)$ is the sum of the logarithms of all primes $\leqq b$,

$$\vartheta(b) \leqq \frac{6}{5}(.92129)b + 3\log^2 b + 8\log b + 5.$$

Take

$$\begin{aligned} b &= (1+n^s)^{2/(s-5)}, \\ -\log c &= \log n + (n+1)\log 3 + 2n^2\log 2 + n(2n-1)\vartheta(b), \\ C &= 12(8n\cdot 3^{n-1})^{1/2}2^{(n-1)/4}n^{3(n-1)/2}/(3/2)^{(1-1/n)/2}, \\ N^{-J/n} &= C/c, \quad 2J = \sigma\left(3-\frac{1}{2n}\right)+z-\frac{(n-1)}{2n^2}, \end{aligned}$$

* Presented to the Society, October 26, 1935.

† Annals of Mathematics, vol. 37 (1936), pp. 293–316, cited as A; American Journal of Mathematics, July, 1936, cited as J.

$$\sigma = n\left(1 - \frac{1}{n}\right)^k, \qquad \begin{array}{l} k > \log r/\{\log n - \log (n-1)\}, \\ r = n^2(6n-1)/(n-1-2n^2z), \end{array}$$

where z is positive and so small that

$$1 + \log (3/2)^{n-1} + (n-1)/(2n^2)\cdot\log N \leqq \left(\frac{1}{3} N^{1/n}\right)^z.$$

3. *Case* $n=6$. The best result is obtained from $s=19$. Take $z=0.00002$. Then the least integer k is 31. Using base 10 except as marked, we get

$$\log \sigma = \bar{2}.3235310, \qquad -J = 0.0039946, \qquad \vartheta(b) = 257.984,$$
$$-\log_e c = 17086.33198, \qquad \log_e C = 2.11363,$$
$$\log\log N = 7.047165.$$

All integers $< l = 2120044$ are sums of 86 sixth powers,* while all between l and $L_0 = 51\ 679\ 845\ 000\ 000$ are sums of 33 sixth powers. Then† all integers between l and L_t are sums of $t+33$ if $\log\log L_t = (0.0791813)t + 0.9563804$. Thus $L_t \geqq N$ if $t=77$. But $t+33=110=s-2+3k$.

THEOREM 1. *Every positive integer is a sum of* 110 *sixth powers.*

4. *Case* $n=11$. Take $z=0.00001$. The least integer k is 70. Also $s \geqq 23$. The best result is obtained from $k=100$, $s=43$. Then

$$\log \sigma = \bar{4}.90212, \qquad -J = 0.0194770, \qquad \vartheta(b) = 388.19352,$$
$$-\log_e c = 89856.02131, \quad \log_e C < \frac{1}{2} n^2, \quad \log\log N = 7.343518.$$

Denote the eleventh powers of $2, \cdots, 9$ by $a, \cdots, h$. By an extended algebraic table (in MS), I proved that every integer between

$$l = 2g + h = 48\ 560\ 928\ 793$$

and $l+15b$ is a sum of 129 eleventh powers. By ascents, all between l and $L_0 = 33\ 348\ 227 \times 10^6$ are sums of 171, and all between l and L_t are sums of $t+171$, where

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* R. C. Shook, University of Chicago Dissertation, 1934.

† This Bulletin, vol. 39 (1933), p. 711, Theorem 12. Here $\nu = 1/6$ to seven decimal places.

$$\log \log L_t = t \log 1.1 + 0.3140358.$$

Hence $L_t \geqq N$ if $t \geqq 170$.

LEMMA 1. *Every integer* $\geqq l$ *is a sum of* 341 *eleventh powers.*

Elsewhere* I proved that 302 eleventh powers suffice from $j = 2d + e = 460\ 453\ 306$ to $j + 14c$ and are necessary for $j + 66a + 8b + 9c + 1416$, while by ascent 321 powers suffice from j to far beyond l.

LEMMA 2. *Every integer* $\geqq j$ *is a sum of* 341 *eleventh powers.*

5. To obtain further results for $n = 11$, employ

(1) $b = 86a + 1019, \qquad c = 23b + 38a + 1139,$

(2) $d = 11c + 15b + 16a + 808, \qquad e = 7d + 5c + 14a - 11.$

My method† to construct an algebraic table of minimum decompositions is based on leaders L. Since L has a decomposition into fewer powers, the same is true of the sum of L and any linear function. The leaders $< e$ are

$173a(=10+2b)$, $64a+165b$, $36a+189b$, $356b$, $253b+c$, $75a+38b+10c$, $86a+27c$, $164b+28c$, $24a+140b+29c$, $35a+13b+46c$, $115b+65c$, $82c$, $126a+d$, $127b+d$, $115a+63b+2d$, $87a+87b+2d$, $40a+25b+3d$.

Employ linear homogeneous functions in which the coefficients of a, b, c, d are $\geqq -86$, -23, -11, -7, respectively, by (1) and (2), and which have no leader as component. We obtain $87a - b = 1029$, 25 equations in a, b, c, 33 with $-d$, 25 with $-2d$, 17 with $-3d$, 14 with $-4d$, 12 with $-5d$, 10 with $-6d$, and 8 with $-7d$.

We shall say that there is a *peak* (x, m), or peak x at m, if m is a sum of x, but not fewer, nth powers, while all integers $> m$ are sums of fewer than x powers. Thus m is the greatest integer requiring x nth powers. If (x', m') and (x, m) are consecutive peaks, every integer between m' and m is a sum of x nth powers.

THEOREM 2. *There are exactly* 24 *peaks* $\geqq 336$ *if* $n = 11$. *They include*

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* Journal of the London Mathematical Society, vol. 9 (1934), pp. 201–206.

† This Bulletin, vol. 40 (1934), pp. 487–493. More details in American Mathematical Monthly, vol. 41 (1934), pp. 547–555.

$P_j=(1185-119(j-1), jc+85a+2047-120j)$, $(j=1, 2, 3)$;
$R_k=(838-119k, (4+k)c+85a+1577-120k)$, $(k=0, 1, 2, 3)$;
$S=(954, 3c+82a+1697)$.

If (x, m) *is any of these peaks except* R_3, *also* $(x-93, m+22b)$ *is a peak. Together, these give the peaks with* $x=1185$, 1092, 1066, 973, 954, 947, 861, 854, 838, 745, 719, 626, 600, 507, 481. *The earlier peaks are* $(2132, u=85a+2047)$, $(1211, u+22\text{b})$. *The later peaks are*

$(454, d+4c+9b+5a+529)$, $(383, 2d+2c+16b+24a+1769)$,
$(373, w+47a+769)$, $(372, w+75a+740)$, $(357, w+83a+740)$,
$(354, 3d+2c+17b+36a+762)$, $(336, 6d+2c+17b+36a+743)$,

where $w=2d+2c+17b$.

6. *Case* $n=9$. Take $z=0.00001$. The least integer k is 54. Also $s\geqq 19$. We take $s=30$, $k=80$, and obtain

$$\log\sigma=\bar{4}.8620425,\qquad -J=0.0236148,\qquad \vartheta(b)=346.26604,$$

$$-\log_e c=53104.1773,\ \log_e C<\frac{1}{2}n^2,\ \log\log N=6.9443023.$$

If $a, \cdots, e$ denote the ninth powers of $2, \cdots, 6$, I proved* that all integers between $h=2d+e$ and $L_0=58221534000$ are sums of 140 ninth powers. By ascent as in §3, all between h and L_t are sums of $t+140$ if

$$\log\log L_t=(0.0511525)t+0.3376543.$$

Then $L_t\geqq N$ if $t=130$. This proves the following lemma.

LEMMA 3. *Every integer* $\geqq h=2d+e$ *is a sum of* 270 *ninth powers.*

THEOREM 3. *There are exactly* 19 *peaks* $\geqq 163$ *if* $n=9$. *If* (x, m) *is one of the four peaks* $(314, c+25a+390)$, $(220, c+12b+25a+284)$, $(207, 3c+12b+25a+269)$, $(194, 5c+12b+25a+254)$, *then* $(x-5, m+12a)$ *is a peak. The further peaks are* $(548, 37a+511)$, $(333, 12b+37a+284)$, $(208, 2c+37a+390)$, $(195, 4c+37a+375)$, $(182, 6c+37a+360)$, $(181, d+6b+25a+220)$, $(177, d+7b+12a+492)$, $(176, d+7b+26a+477)$, $(175, d+8b+2a+235)$, $(169, d+8b+11a+220)$, $(163, 2d+11b+12a+473)$.

* This Bulletin, vol. 40 (1934), pp. 487–493.

The first four peaks give the greatest integers requiring 548, 333, 314, 309 ninth powers, respectively. There is very strong evidence that a like result holds for the next 15 peaks. For example, all integers between $e+d$ and $e+2d$ are sums of 128 ninth powers; all between $e+2d$ and $e+3d$ are sums of 125.

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MAPS OF ABSTRACT TOPOLOGICAL SPACES IN BANACH SPACES*

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1. *Introduction.* This paper is to serve as a brief introduction to the method of considering the analysis of abstract topological spaces through the medium of homeomorphic mappings of these spaces on subsets of Banach spaces.† Our primary objective here, however, is to obtain for some general topological groups the abstract correspondents of the fundamental Lie partial differential equations for an r-parameter continuous group.‡ The essential notion is the treatment of the general situation with the aid of abstract *coordinates* in Banach spaces wherein the Fréchet differential may be used.§

By an abstract topological space is meant here a set of elements of completely unspecified nature, together with an undefined concept, that of neighborhood of an element (we denote the elements by small Latin letters, and the neighborhood associated with an element a by $U(a)$), satisfying the four Hausdorff postulates given below.‖

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* Presented to the Society, November 30, 1935.

† S. Banach, *Théorie des Opérations Linéaires*, 1932.

‡ S. Lie, *Theorie der Transformationsgruppen*, vols. 1, 3.

§ M. Fréchet, *Annales de l'École Normale Supérieure*, (3), vol. 42 (1925), p. 293. Briefly, $f(x)$ on B_1 to B_2 has a differential at $x=x_0$, if there exists a function $f(x; z)$ on B_1^2 to B_2, linear (additive and continuous) in z and such that given a $\rho>0$ there is determined a $n>0$, so that $\|f(x_0+z)-f(x_0)-f(x_0; z)\| \leqq \rho\|z\|$ for $\|z\|\leqq n(\rho)$; $f(x_0; z)$ is the differential. See also various papers by Hildebrandt, Graves, Kerner, Michal, and many others.

‖ F. Hausdorff, *Mengenlehre*, 1927, pp. 226–229.