NORMALS TO A SPACE V_n IN HYPERSPACE

BY NOLA L. ANDERSON AND LOUIS INGOLD

1. *Introduction.* The authors consider here a generalization of known results relating to the curvature vectors of a pair of mutually orthogonal curve systems on a general two-dimensional surface S_2 in hyperspace. Several phases of this generalization have been given in papers presented from time to time* but the more connected account here given seems desirable. The generalization is of a two-fold nature: first, to vector systems, or curve systems, in n dimensions, and second, to systems not necessarily orthogonal.†

The results have been presented as they are related to the usual type of n-dimensional geometry, namely, a geometry in which tangent vectors f_i to curves of parameter x^i may be obtained by differentiation from a fundamental vector or function f. The fundamental tensor g_{ij} is determined by the formulas $f_i \cdot f_j = g_{ij}$, where the product on the left is a scalar product if a vector notation is used, or an integrated product if the function notation is employed. The discriminant $|g_{ij}|$ is assumed to be different from zero.

In the present paper, certain abbreviations are used, but the notations are essentially the same as in the paper just cited. There is also a slightly wider interpretation given to the Maschke parenthesis expressions. Thus in the vector $(fa^1 \cdots a^{n-1})$ the a^k

---

* The paper in the form here offered is the outgrowth of work begun in two papers presented to the Society: *First normal spaces in Riemannian geometry*, by Nola L. Anderson, presented at Lawrence, Kansas, December 1, 1928, and *Invariant normals to a space S_n contained in a function space*, by Nola L. Anderson and Louis Ingold, presented at Des Moines, Iowa, December 31, 1929. All the results had been secured and the general organization had been discussed, before the sad death of Professor Ingold on January 25, 1935 (see this Bulletin, vol. 41, p. 181).

† A brief discussion of these ideas for orthogonal systems was given by L. Ingold in the paper *A symbolic treatment of the geometry of hyperspace*, Transactions of this Society, vol. 27, pp. 574–599, but the treatment was inadequate because of the necessity of restricting the discussion to a fixed point of the space. Those unfamiliar with the methods and notations may be referred to this paper, where other references will be found.

are regarded as arbitrary covariant vectors or vector fields of components a_i^k, and the parenthesis denotes $1/g^{1/2}$ times the determinant whose first column consists of the f_i and whose column headed a^k consists of the a_i^k. The more usual case in which the a^k denote functions of the coordinates and in which a_i^k denotes $\partial a^k/\partial x^i$ is of course a special case.

When n covariant vectors $a^1, a^2, \cdots, a^n$ are used, the vectors $(fa^1 \cdots a^{i-1}a^{i+1} \cdots a^n)$ will be written in the abbreviated form (fA_i). Other abbreviations will generally be clear from these two illustrations* and the context. Where other equivalent notations for the vector or symbol f are used in parenthesis expressions, the context will show how many notations of the different types are present. The expression $(f\phi)(\phi A_i)$, where the f's and ϕ's denote equivalent notations for the fundamental function or vector f, must contain just one notation ϕ and $n-1$ notations f, since the A_i stands for $n-1$ columns. The unabbreviated form would be

$$(f^1f^2 \cdots f^{n-1}\phi)(\phi a^1a^2 \cdots a^{i-1}a^{i+1} \cdots a^n).$$

The expressions

$$(f^1 \cdots f^ka^1 \cdots a^{n-k}) \qquad \text{and} \qquad (f^1 \cdots f^kb^1 \cdots b^{n-k})$$

may be written simply (fa) and (fb), where the context shows how many of the a's and b's are present. The angle θ between the two k-dimensional forms just given is defined by the formula†

$$\cos\theta = \frac{(fa)(fb)}{[(\phi a)^2(\psi b)^2]^{1/2}}.$$

The use of distinct notations of ϕ, ψ instead of f in the denominator is not strictly necessary here.

We need only the angles ω_{ij} between the vectors (fA_i) and (fA_j), and the angles α_{ij} between the $(n-1)$-dimensional vectors $(f^1 \cdots f^{n-1}a^i)$ and $(f^1 \cdots f^{n-1}a^j)$.

It will be convenient also to introduce notations for certain frequently occurring invariants. Thus S will denote the paren-

* The letters $f, \phi, \psi, \theta, \cdots$, with upper indices where necessary, will be used as equivalent notations for the fundamental function or vector.

† See Anderson, *The trigonometry of hyperspace*, The American Mathematical Monthly, vol. 36, pp. 517–523.

thesis $(a^1a^2 \cdots a^n)$ divided by the product of factors of the form $[(a^m\psi)^2/(n-1)!]^{1/2}$ for $m=1, 2, \cdots, n$. In some of the formulas a second set of a's: $a'^1, a'^2, \cdots, a'^n$ are used, and S' is formed from these as S above is formed from $a^1, a^2, \cdots, a^n$.

S_r will be used to denote $[(\phi A_r)^2]^{1/2}$ divided by the product of factors of the form $[(a^m\phi)^2/(n-1)!]^{1/2}$ except that the one factor $[(a^r\phi)^2/(n-1)!]^{1/2}$ for which $m=r$ is to be omitted.

The invariant corresponding to S constructed from

$$a^1 \cdots a^{r-1}a'^m a^{r+1} \cdots a^n$$

will be denoted by $S_{m')r}$.

The invariant corresponding to S_m but constructed wholly from the $a'^1, \cdots, a'^n$ will be denoted by S_m'. These invariants are easily expressed in terms of trigonometric functions in the two and three dimensional cases.*

2. *Curvature Vectors*. Let there be given any vector field $\boldsymbol{a}$, that is, a vector $\boldsymbol{a}$ which is a function of the coordinates x^i. Also consider a set of vectors (fA_i) constructed from the covariant set a_m^k as described in the introduction. Let $\bar{\boldsymbol{a}}$ denote $\boldsymbol{a}/(\boldsymbol{a}\cdot\boldsymbol{a})^{1/2}$, so that $\bar{\boldsymbol{a}}$ is a unit vector, and let s_i denote arc length along any curve having the vector (fA_i) for a tangent vector; then $d\bar{\boldsymbol{a}}/ds_i$ will be called the curvature vector of the vector $\boldsymbol{a}$ with respect to s_i. It is evident that the curvature of $\boldsymbol{a}$ with respect to s_i is the same as the curvature of $h\boldsymbol{a}$, where h is any scalar.

The vectors $(f\phi)(\theta\phi)(\theta A_i)$ have the same direction as the vectors (fA_i), so that the curvatures of these two with respect to any direction are the same. We wish to compute the components of the curvature vectors of (fA_i) with respect to s_j which are orthogonal to all of the (fA_i), that is, normal to the space V_n. The unit vector in the direction of $(f\phi)(\theta\phi)(\theta A_i)$ is obtained by dividing by its magnitude $(n-1)![(\theta A_i)^2]^{1/2}$, and the derivative of this unit vector with respect to s_j consists of several terms all of which are tangent to V_n except

$$N_{ij} = \frac{(\theta\phi)(\theta A_i)((f\phi)A_j)}{(n-1)![(\theta A_i)^2(\theta A_j)^2]^{1/2}},$$

* They may also be interpreted trigonometrically in space of higher number of dimensions.

and these are clearly normal to V_n. They are therefore the normal components required.

We may consider in a similar way the normal components of the curvatures of the vectors $(f\phi)(\phi a^i)$ with respect to s_j. It is easily shown that the product $(f\phi)(\phi a^i)(fa^ib^1 \cdots b^{n-2})=0$, where the b_m^k are the components of covariant vectors for each k, but since $(f\phi)(\phi a^i)$ belongs to V_n we say that it is a *space* normal to all of the vectors $(fa^ib^1 \cdots b^{n-2})$. When a^i is a function of the coordinates and $a_m^i=\partial a^i/\partial x^m$, the vector $(f\phi)(\phi a^i)$ is a normal in V_n to the hypersurface $a^i=$constant. If we proceed as above, we obtain the normals*

$$\mathcal{N}_{ij} = \frac{((f\phi)A_j)(a^i\phi)}{[(n-1)!(a^i\psi)^2(\theta A_j)^2]^{1/2}}.$$

The vectors N_{ij} and $\mathcal{N}_{ij}$ will be referred to as the first and second set, respectively, of normal curvatures of the system of covariant vectors a^i. It is readily seen that $N_{ij}=N_{ji}$; however, $\mathcal{N}_{ij}\neq\mathcal{N}_{ji}$ in general, but still certain relations must exist among the normal curvatures of the second set because, as we shall show, they are linearly expressible in terms of those of the first set.

3. *Relations among the Normal Curvatures.* Introducing the factor $(a^1a^2 \cdots a^n)\equiv(a)$ in both numerator and denominator of the expression for N_{ij}; we have

$$N_{ij} = \frac{(a)(\theta\phi)(\theta A_i)((f\phi)A_j)}{(a)(n-1)![(\theta A_i)^2(\theta A_j)^2]^{1/2}}.$$

By use of the identity

$$(a)(\theta\phi) = \sum_{r=1}^{r=n} (a^1 \cdots a^{r-1}\theta a^{r+1} \cdots a^n)(a^r\phi),$$

this reduces after some rearrangement to

$$N_{ij} = \sum_{r=1}^{r=n} \frac{(-1)^{r+1}(\theta A_r)(\theta A_i)(a^r\phi)((f\phi)A_j)}{(n-1)![(\theta A_i)^2(\theta A_j)^2]^{1/2}},$$

* Other normal curvatures, of course exist. If arc lengths of curves tangent to the vectors $(f\phi)(\phi a^i)$ are denoted by σ_i, the curvature of these vectors with respect to σ_j is clearly another example, and undoubtedly interesting relations involving these curvatures analogous to those given in the text would exist.

and this is readily reduced to

$$N_{ij} = \sum_{r=1}^{r=n} (-1)^{r+1} \frac{S_r}{S} \cos \omega_{ri} \mathcal{N}_{rj}.$$

In order to express, conversely, the $\mathcal{N}_{ij}$ in terms of the N_{ij}, we make use of the identity

$$(f\phi)(a^i\phi)(a) = \sum_{r=1}^{r=n} (-1)^{r+1}(a^i\phi)(a^r\phi)(fA_r),$$

which gives

$$\frac{(f\phi)(a^i\phi)}{[(n-1)!(a^i\phi)^2]^{1/2}} = \sum_{r=1}^{r=n} \frac{(-1)^{r+1}(a^i\phi)(a^r\phi)[(\theta A_r)^2]^{1/2}}{(a)[(n-1)!(a^i\psi)^2]^{1/2}} \cdot \frac{(fA_r)}{[(\theta A_r)^2]^{1/2}}.$$

The components normal to V_n of the derivatives with respect to s_j of the two sides of this equation are equal; hence

$$\mathcal{N}_{ij} = \sum_{r=1}^{r=n} \frac{(-1)^{r+1}(a^i\phi)(a^r\phi)[(\theta A_r)^2]^{1/2}}{(a)[(n-1)!(a^i\psi)^2]^{1/2}} N_{rj},$$

and this readily reduces to

$$\mathcal{N}_{ij} = \sum_{r=1}^{r=n} (-1)^{r+1} \frac{S_r}{S} \cos \alpha_{ri} N_{rj}.$$

In two dimensions consider the system of curves $a^i = \text{const}$. At each point this system consists of just one pair of curves. There is a single angle $\omega_{12} = \alpha_{12} = \omega$; of course, $\omega_{11} = \omega_{22} = 0$. The two sets of formulas for N_{ij} and $\mathcal{N}_{ij}$ are

$$N_{11} = \frac{\mathcal{N}_{11} - \cos \omega \mathcal{N}_{21}}{\sin \omega}, \qquad \mathcal{N}_{11} = \frac{N_{11} - \cos \omega N_{21}}{\sin \omega},$$

$$N_{12} = N_{21} = \frac{\mathcal{N}_{12} - \cos \omega \mathcal{N}_{22}}{\sin \omega} \qquad \mathcal{N}_{12} = \frac{N_{12} - \cos \omega N_{22}}{\sin \omega},$$

$$= \frac{\cos \omega \mathcal{N}_{11} - \mathcal{N}_{21}}{\sin \omega}, \qquad \mathcal{N}_{21} = \frac{\cos \omega N_{11} - N_{21}}{\sin \omega},$$

$$N_{22} = \frac{\cos \omega \mathcal{N}_{12} - \mathcal{N}_{22}}{\sin \omega}, \qquad \mathcal{N}_{22} = \frac{\cos \omega N_{12} - N_{22}}{\sin \omega}.$$

The following are sample formulas in three dimensions:

$$N_{11} = (1/S)(\sin A\, \mathcal{N}_{11} - \sin B \cos \gamma\, \mathcal{N}_{21} + \sin C \cos \beta\, \mathcal{N}_{31}),$$

$$\mathcal{N}_{11} = (1/S)(\sin A N_{11} - \sin B \cos C N_{21} + \sin C \cos B N_{31}).$$

4. *Normal Curvatures of two Systems.* In this section we obtain formulas connecting the normal curvatures of the first set belonging to a system a^i with those belonging to another system a'^i. We multiply numerator and denominator of the expressions for N_{ij} by the square of $(a'^1 a'^2 \cdots a'^n)$,

$$N_{ij} = \frac{(\theta\phi)(\theta A_i)((f\phi)A_j)(a')^2}{(a')^2\left[(\theta A_i)^2(\theta A_j)^2\right]^{1/2}(n-1)!}.$$

By making use of the identities

$$((f\phi)A_j)(a') = \sum_{r=1}^{r=n} (-1)^{r+1}(a'^r A_j)((f\phi)A'_r),$$

$$(\theta A_i)(a') = \sum_{m=1}^{m=n} (-1)^{m+1}(a'^m A_i)(\theta A'_m),$$

we have

$$\begin{aligned} N_{ij} &= \sum_{r=1}^{r=n} \sum_{m=1}^{m=n} \frac{(-1)^{r+m}(\theta\phi)(a'^r A_j)(a'^m A_i)(\theta A'_m)((f\phi)A'_r)}{(a')^2\left[(\theta A_i)^2(\theta A_j)^2\right]^{1/2}(n-1)!} \\ &= \sum_{r=1}^{r=n} \sum_{m=1}^{m=n} \frac{(-1)^{r+m}(a'^r A_j)(a'^m A_i)\left[(\theta A'_m)^2(\theta A'_r)^2\right]^{1/2}}{(a')^2\left[(\theta A_i)^2(\theta A_j)^2\right]^{1/2}} N'_{rm} \\ &= \sum_{r=1}^{r=n} \sum_{m=1}^{m=n} (-1)^{r+m} \frac{S_{r'})_j S_{m'})_i S'_m S'_r}{S'^2 S_i S_j} N'_{rm}. \end{aligned}$$

This is the formula desired. It is a generalization of the formulas connecting the normal curvatures of two orthogonal families of curves on a two dimensional surface with those of a second orthogonal family. It is a generalization not only to n dimensions, but also to systems not necessarily orthogonal. A similar procedure leads to the formula

$$\mathcal{N}_{ij} = (-1)^{r+m} \frac{S_{r'})_j S'_{i)m'} S'_r}{S_j S'^2} \mathcal{N}'_{mr}.$$

Consider the case of two curves on a surface, $a=$const., $b=$const., meeting at an angle ω, and let $a'=$const. pass through their point of intersection P making an angle α with $a=$const.

Also let b' = const. pass through P making an angle β with a = const. Then we may write

$$N'_{12} = \frac{\sin(\omega - \alpha)\sin(\omega - \beta)}{\sin^2 \omega} N_{11}$$
$$+ \frac{\sin\alpha\sin(\omega - \beta) + \sin\beta\sin(\omega - \alpha)}{\sin^2 \omega} N_{12} + \frac{\sin\alpha\sin\beta}{\sin^2\omega} N_{22}.$$

This formula gives the normal curvatures of any two curves through P in terms of the normal curvatures of any given pair of curves a = const., b = const., and the angles which the different curves make with a = const. The same formula takes care of all cases. Thus to obtain N'_{11} make $\beta = \alpha$; to obtain N'_{22} make $\beta = \alpha$.

To obtain the usual formulas for two orthogonal systems make $\omega = \pi/2$, and then in turn (1) $\beta = \alpha$, (2) $\beta = \alpha + \pi/2$, (3) place $\beta = \alpha$ and then replace α by $\alpha + \pi/2$. We now have the following known formulas.*

(1) $$N'_{11} = \cos^2 \alpha N_{11} + 2 \sin\alpha \cos\alpha N_{12} + \sin^2 \alpha N_{22},$$

(2) $$N'_{12} = \sin\alpha\cos\alpha(N_{22} - N_{11}) + (\cos^2\alpha - \sin^2\alpha)N_{12},$$

(3) $$N'_{22} = \sin^2\alpha N_{11} - 2\sin\alpha\cos\alpha N_{12} + \cos^2\alpha N_{22}.$$

We mention finally that in this general two-dimensional formula for N'_{12}, if α is kept constant while β varies, the formula

$$N'_{12} = \frac{\sin(\omega - \beta)}{\sin^2\omega}[\sin(\omega - \alpha)N_{11} + \sin\alpha N_{12}]$$
$$+ \frac{\sin\beta}{\sin^2\omega}[\sin(\omega - \alpha)N_{12} + \sin\alpha N_{22}]$$

shows that N_{12} lies in the plane of the two fixed vectors

$$\sin(\omega - \alpha)N_{11} + \sin\alpha N_{12} \quad \text{and} \quad \sin(\omega - \alpha)N_{12} + \sin\alpha N_{22}.$$

NEWCOMB COLLEGE, TULANE UNIVERSITY, AND
THE UNIVERSITY OF MISSOURI

---

* See Wilson and Moore, *Differential geometry of two dimensional surfaces in hyperspace*, Proceedings of the American Academy of Arts and Sciences, vol. 52 (1916), pp. 270–364.