A THEOREM CONCERNING LOCALLY PERIPHERALLY SEPARABLE SPACES*

BY F. B. JONES

Alexandroff has shown that a connected metric space is completely separable provided it is locally completely separable.† It is the purpose of this paper to establish a similar theorem for connected, locally connected metric spaces.

DEFINITION. *A space is locally peripherally separable provided that, if P is a point of a region R, there exists in R a domain D containing P such that the boundary of D is separable.*

THEOREM. *A connected, locally connected, locally peripherally separable metric space is completely separable.*

PROOF. Suppose that n is a fixed positive integer. Let G denote the collection of all domains (to be called regions) of diameter $1/n$ or less which are peripherally separable. Since space is locally peripherally separable, it is evident that G covers space. It will now be shown that G contains a countable subcollection covering space. For each point X of space let n_x denote the largest integer such that no region of G contains a circular domain with center at X and radius greater than or equal to $1/n_x$. Let D_1 denote some region of G. For each integer i let M_{1i} denote the set of all points X of the boundary β_1 of D_1 such that $n_x = i$. Since space is metric and β_1 is separable , β_1 is completely separable, and there exists in M_{1i} a countable point set N_{1i} which is everywhere dense in M_{1i}. Now for each point X of N_{1i} let R_x denote a region of G containing a circular domain with center at X and radius $1/(i+1)$. The sum of these regions R_x forms a domain Q_{1i} covering M_{1i}, and $\sum_{i=1}^{\infty} Q_{1i}$ is a domain covering β_1.

Let $D_2 = D_1 + \sum_{i=1}^{\infty} Q_{1i}$. Then D_2 contains $D_1 + \beta_1$. Since space is locally connected, every point of the boundary β_2 of D_2 either belongs to the boundary of some region R_x or is a limit point of

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† Paul Alexandroff, *Über die Metrization der im kleinen kompakten topologischen Räume*, Mathematische Annalen, vol. 92 (1924), pp. 294–301.

the sum of the boundaries of these regions R_x.* Furthermore, there are only countably many regions R_x. Hence, β_2 is separable. It is also to be noted that D_2 is the sum of countably many regions of G.

This process may be continued. Thus there exist a sequence $D_1, D_2, D_3, \cdots$ of domains and for each integer m a sequence $N_{m1}, N_{m2}, N_{m3}, \cdots$ of point sets, so that if, for each integer m, β_m denotes the boundary of D_m, and, for each pair of integers m and i, M_{mi} denotes the set of all points X of β_m such that $n_x = i$, then

(0) D_m is the sum of countably many regions of G,

(1) D_m contains $D_{m-1}+\beta_{m-1}$,

(2) $\beta_m = \sum_{i=1}^{\infty} M_{mi}$,

(3) N_{mi} is a countable subset of M_{mi} which is everywhere dense in M_{mi},

(4) if X is a point of N_{mi}, D_{m+1} contains a circular domain with center at X and radius $1/(i+1)$, and

(5) if R is a region of G containing a point X of N_{mi}, no circular domain V_x lying in R with center at X has a radius greater than or equal to $1/i$.

Suppose now that there is some point of space not belonging to any D_m. Then, since space is connected, there exists a point O on the boundary of $\sum D_m$. Hence, it follows from (1), (2), and (3) that O is a sequential limit point of a sequence of points $X_1, X_2, X_3, \cdots$, where for each m, X_m belongs to N_{mi} for some i. Now, let R denote a region of G containing a circular domain V_0 with center at O and radius r. For each point X_m of the sequence $X_1, X_2, X_3, \cdots$ let V_{X_m} denote a circular domain with center at X_m and radius $r/2$. There exists a number k such that if $m>k$, then V_{X_m} lies in R. However, since O is not covered by $\sum D_m$ and N_{mi} contains X_m, it follows from (4) that i increases with m. Hence, for some point X_m, $(m>k)$, $r/2$ is greater than $1/i$. This contradicts (5); so $\sum D_m$ covers space. By (0), D_m is the sum of countably many regions of G, and therefore $\sum D_m$ is the sum of the regions of a countable subcollection G' of G. Hence G' covers space.

Now, for each integer n, let G_n denote the collection of periph-

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* R. L. Moore, *Foundations of Point Set Theory*, Colloquium Publications of this Society, vol. 13, 1932, Theorem 2(b), Chapter II, p. 91.

erally separable domains of diameter $1/n$ or less. By the above argument each G_n contains a countable subcollection G_n' covering space. Hence, space is completely separable and the theorem is established.

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A REDUCED SET OF POSTULATES FOR ABSTRACT HILBERT SPACE*

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1. *Introduction.* An abstract Hilbert space is a normed linear space, or *vector space*, of infinite dimensionality, with a norm based on a Hermitian *inner product*, defined for all pairs of elements in the space. The space is, moreover, separable and complete according to this norm. The usual postulate system for Hilbert space, which was first stated abstractly by J. von Neumann, consists of five groups of postulates, or nineteen in all.†

The purpose of the present paper is to demonstrate the redundancy‡ of a number of the postulates, and to present a system of eleven independent postulates for a normed linear space with a Hermitian inner product. The adjunction of three more postulates, each of which is independent of the first eleven and the remaining two, then gives us a system which is equivalent to that of von Neumann, that is, it defines an abstract Hilbert space, and it is categorical.

A special feature of this postulate system is that the abstract relation called equality, and denoted, as usual, by the symbol =, enters on an equal footing with the operations defined in the space.§ Three of the eleven postulates are concerned with this

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† J. von Neumann, *Mathematische Grundlagen der Quantenmechanik*, 1932. pp. 19–24; M. H. Stone, *Linear Transformations in Hilbert Space*, Colloquium Publications of this Society, vol. 15, pp. 2–4.

‡ Some of these redundancies were noted simultaneously by a fellow-student, Mr. Ivar Highberg, and myself.

§ The postulational treatment of equality in vector spaces was suggested by A. D. Michal in a critique of postulate systems. See this Bulletin, vol. 39 (1933), Abstract No. 339.