

statement is obviously true; if  $U_1$  lies in  $[U_1]$ ,  $A = U_1 U_1^* P_1 U_1$  and, since  $U_1^* P_1 U_1$  is a positive hermitian matrix of rank  $r$ ,  $U_1^* P_1 U_1 = P_2$  and  $U_1$  lies in  $[U_2]$ . Similarly any member  $U_2$  of  $[U_2]$  lies in  $[U_1]$ . Further the matrix  $P_2$  is invariant under unitary transformation by any matrix of the group  $G_1$ , and  $P_1$  under transformation by any matrix of the group  $G_2$ . For if  $Z_1$  lies in  $G_1$ ,  $A Z_1 = A$  so that  $A = U_2 Z_1 Z_1^* P_2 Z_1$ , and accordingly,  $Z_1^* P_2 Z_1 = P_2$ .

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### ON A THEOREM OF FÉRAUD

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The Birkhoff-Pfaffian equations of dynamics are written in variational form as follows:

$$\delta \int \left[ \sum_{i=1}^{2m} X_i \left( \frac{dx_i}{dt} \right) + Q \right] dt = 0,$$

where  $Q$  and the  $X$ 's are functions of  $x_1, \dots, x_{2m}$  and, in general, depend also periodically upon  $t$ , and where the skew-symmetric determinant  $|a_{ij}|$ , ( $a_{ij} = \partial X_i / \partial x_j - \partial X_j / \partial x_i$ ), does not vanish in the regions considered. We restrict attention to the neighborhood of a generalized equilibrium point, that is, a point where all the  $\partial Q / \partial x_i - \partial X_i / \partial t$  vanish identically in  $t$ . We take this point at the origin,  $x_i = 0$ , ( $i = 1, 2, \dots, 2m$ ).

The problem of reducing the Pfaffian system to a Hamiltonian system can be reduced to that of finding a non-singular transformation,  $x_i = x_i(y_1, \dots, y_{2m})$ , leaving the origin invariant (and depending in general periodically upon  $t$ ) which reduces the linear differential form  $\sum_{i=1}^{2m} X_i dx_i$  to the form  $\sum_{i=1}^m y_{2i} dy_{2i-1} + dw$ , where  $dw$  is an exact differential in  $y_1, \dots, y_{2m}$ , the coefficients of which are independent of  $t$ . This same problem also will play an important role in a future paper of mine on "conservative" transformations in  $2m$ -dimensional spaces.

The problem has been considered by Féraud,† who obtained a

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† *Extension au cas d'un nombre quelconque de degrés de liberté d'une propriété relative aux systèmes Pfaffiens*, Comptes Rendus, vol. 190 (1930), pp. 358-360.

solution based on the fundamental existence theorem of Riquier for systems of partial differential equations. It appears, however, highly desirable to give a more elementary proof based on existence theorems for *ordinary* differential equations. The result is thus not limited to the analytic case, as it was by the use of Riquier's theorem in Féraud's proof.

Our proof follows a well known line of thought expounded by Goursat.\* There are several reasons, however, why Goursat's writings on the reduction of linear differential forms are not immediately applicable. For one thing, the transformations used by him do not necessarily leave the origin invariant. Also there are important cases where his reasoning might fail.† The present treatment does not presuppose a knowledge of Goursat's work.

We split up our proof into a number of easy lemmas.

LEMMA 1. Consider a linear differential form  $\sum_{i=1}^n X_i dx_i$  which under a non-singular transformation of coordinates,  $x_i = g_i(y_1, \dots, y_n)$ , appears also in the form  $\sum_{i=1}^n Y_i dy_i$ . Then we have the following elementary relations:

$$(1) \quad Y_i = \sum_{j=1}^n X_j \left( \frac{\partial g_j}{\partial y_i} \right),$$

$$(2) \quad \frac{\partial Y_i}{\partial y_r} - \frac{\partial Y_r}{\partial y_i} = \sum_{l,i,j=1}^n \left( \frac{\partial X_j}{\partial x_l} - \frac{\partial X_l}{\partial x_j} \right) \frac{\partial g_l}{\partial y_r} \frac{\partial g_i}{\partial y_j}.$$

The proof of (1) is trivial, while (2) is deduced at once from (1) by differentiation with respect to  $y_r$ , interchange of indices, and subtraction. In the sequel, we shall uniformly set  $a_{ij} = \partial X_i / \partial x_j - \partial X_j / \partial x_i$ , and  $b_{ij} = \partial Y_i / \partial y_j - \partial Y_j / \partial y_i$ ; so that (2) may also be written in the form

$$(3) \quad b_{ij} = \sum_{k,l=1}^n a_{kl} \frac{\partial g_k}{\partial y_i} \frac{\partial g_l}{\partial y_j}.$$

\* *Leçons sur le Problème de Pfaff*, 1922, especially Chapter 1.

† For example, if the first method of Chapter I is used, care must be taken, in the series of necessary reductions, that at each step the  $\lambda_i$  (pages 11 and 12) do not all vanish at the origin. The present paper does, in effect, just this. If the more advanced method of Chapter 4 is used, it would be necessary to compute the derived forms. Also the most available results of Chapter 4 are stated only for the analytic case.

LEMMA 2. Consider the system of ordinary differential equations

$$\frac{dx_i}{dy_n} = f_i(x_1, \dots, x_n), \quad (i = 1, 2, \dots, n),$$

where the  $f_i$  are defined and of class  $C^v$  in the neighborhood of the origin. Let  $g_i(y_1, \dots, y_{n-1}, y_n)$  be the solution which satisfies the initial conditions  $x_1 = y_1, \dots, x_{n-1} = y_{n-1}, x_n = 0$  for  $y_n = 0$ . Then the  $g$ 's are of class  $C^v$  in the neighborhood of the origin, and, if the equations  $x_i = g_i(y_1, \dots, y_{n-1}, y_n)$  are thought of as defining a transformation  $T$  of the neighborhood of the origin into itself, then  $T$  is non-singular if  $f_n(0, \dots, 0) \neq 0$ .

The fact that the  $g$ 's are of class  $C^v$  is an immediate consequence of a well known theorem.\* The rest of the lemma follows from the obvious relations

$$(4) \quad \begin{aligned} \frac{\partial g_i}{\partial y_j} \Big|_0 &= \delta_{ij}, & (i = 1, 2, \dots, n), \\ \frac{\partial g_i}{\partial y_j} \Big|_0 &= f_i(0, \dots, 0), & (j = 1, 2, \dots, n-1). \end{aligned}$$

For the jacobian of  $T$ , evaluated at the origin, turns out to be exactly  $f_n(0, \dots, 0)$ .

LEMMA 3. Consider a Pfaffian  $\omega = \sum_{i=1}^{2k} X_i dx_i$  satisfying the following conditions:

- (A) The  $X$ 's are of class  $C^\mu$  in the neighborhood of the origin.
- (B) None of the  $X$ 's vanish at the origin.
- (C) The skew-symmetric determinant  $|a_{ij}|$  does not vanish at the origin.

Then there exists a non-singular transformation  $x_i = g_i(y_1, \dots, y_{2k})$  of the neighborhood of the origin into itself such that  $\omega$  appears expressed in terms of the  $y$ 's in the form  $e^{y_{2k}} [\sum_{i=1}^{2k-1} A_i dy_i]$ , where the  $A$ 's are functions of  $y_1, \dots, y_{2k-1}$  satisfying the following conditions:

- ( $\alpha$ ) The  $A_i$  are of class  $C^{\mu-2}$ .
- ( $\beta$ ) None of the  $A_i$  vanish at the origin.

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\* See G. A. Bliss, *The solutions of differential equations of the first order as functions of their initial values*, Annals of Mathematics, vol. 6 (1904-1905), pp. 49-68, especially p. 67.

( $\gamma$ ) The skew-symmetric matrix  $(c_{ij})$ , where  $c_{ij} = \partial A_i / \partial y_j - \partial A_j / \partial y_i$ , is such that there exists a system of functions  $\xi_i(y_1, \dots, y_{2k-1})$  which satisfy the linear identities  $\sum_{j=1}^{2k-1} c_{ij} \xi_j \equiv 0$  and  $\sum_{j=1}^{2k-1} A_j \xi_j \equiv 1$ .

PROOF. Consider the equations

$$(5) \quad \sum_{j=1}^{2k} a_{ij} \left( \frac{dx_j}{dy_{2k}} \right) = X_i, \quad (i = 1, 2, \dots, 2k).$$

Since  $|a_{ij}| \neq 0$ , we may solve for the derivatives, writing the result in the form  $dx_i/dy_{2k} = f_i(x_1, \dots, x_{2k})$ , where the  $f$ 's are of class  $C^{\mu-1}$ . On account of (5) and (B) it is seen that not all the  $f$ 's vanish at the origin. It is no loss of generality to assume  $f_{2k}(0, \dots, 0) \neq 0$ , since a preliminary linear transformation can always be applied, the effect of which is merely to interchange indices in the desired manner. We now apply Lemma 2 to the present system, taking  $n = 2k$  and  $\nu = \mu - 1$ . We proceed to show that the functions  $g_i$ , as there defined, satisfy the conditions of the present lemma.

In equations (5) replace  $dx_i/dy_{2k}$  by  $\partial g_i / \partial y_{2k}$ . Multiply each equation by  $\partial g_i / \partial y_{2k}$  and sum with respect to  $i$ . Since  $a_{ij} + a_{ji} = 0$ , the result is  $\sum X_i (\partial g_i / \partial y_{2k}) = 0$ . Hence from (1) we see that

$$(6) \quad Y_{2k} \equiv 0.$$

We also have, of course,  $\partial Y_{2k} / \partial y_i = 0$ , so that (2) yields

$$\frac{\partial Y_i}{\partial y_{2k}} = \sum_{l,j=1} a_{jl} \frac{\partial g_l}{\partial y_{2k}} \frac{\partial g_j}{\partial y_i},$$

and, since the  $g$ 's satisfy (5), the right member is equal to  $\sum X_j (\partial g_j / \partial y_i)$ , which by (1) is precisely  $Y_i$ . Hence we have the result that  $\partial Y_i / \partial y_{2k} \equiv Y_i$ , or

$$(7) \quad Y_i = e^{\nu_{2k}} A_i,$$

where  $A_i$  is a suitably chosen function of  $y_1, \dots, y_{2k-1}$ , but independent of  $y_{2k}$ . From (6) and (7) we have

$$\omega = \sum_{i=1}^{2k} Y_i dy_i = e^{\nu_{2k}} \left[ \sum_{i=1}^{2k-1} A_i dy_i \right].$$

Now the  $g$ 's are of class  $C^{\mu-1}$  by Lemma 2. Hence the  $Y$ 's and, therefore, the  $A$ 's, are of class  $C^{\mu-2}$  by (1). By (7), (1), and (4),

we have  $A_i = Y_i = \sum_{j+1}^{2k} X_j \delta_{ji} = X_i$ , ( $i = 1, 2, \dots, 2k-1$ ), at the origin. Hence ( $\beta$ ) follows at once from (B).

It remains to show that the  $A$ 's as defined above satisfy (C). It follows from (3) that

$$|b_{ij}| = |a_{ij}| \cdot \left( \frac{\partial(g_1, \dots, g_{2k})}{\partial(y_1, \dots, y_{2k})} \right)^2 \neq 0.$$

Hence we have a unique set of functions  $\xi_1, \dots, \xi_{2k}$  satisfying the equations

$$\sum_{j=1}^{2k-1} b_{ij} \xi_j + b_{i,2k} \xi_{2k} = 0, \quad (i = 1, 2, \dots, 2k-1),$$

$$\sum_{j=1}^{2k-1} -b_{j,2k} \xi_j + 0 = -e^{y_{2k}}, \quad (\text{since } b_{2k,j} = -b_{j,2k}).$$

Now

$$b_{j,2k} = \frac{\partial Y_j}{\partial y_{2k}} - \frac{\partial Y_{2k}}{\partial y_j} = Y_j = e^{y_{2k}} A_j, \quad (j = 1, 2, \dots, 2k-1),$$

and

$$b_{ij} = e^{y_{2k}} \left( \frac{\partial A_i}{\partial y_j} - \frac{\partial A_j}{\partial y_i} \right) = e^{y_{2k}} c_{ij}, \quad (i, j = 1, 2, \dots, 2k-1),$$

on account of (6) and (7). After dividing through by  $e^{y_{2k}}$ , our system of  $2k$  linear equations may, therefore, be written in the form,

$$\sum_{j=1}^{2k-1} c_{ij} \xi_j + A_i \xi_{2k} = 0, \quad \sum_{j=1}^{2k-1} A_j \xi_j = 1.$$

The coefficients now are independent of  $y_{2k}$ . Furthermore  $\xi_{2k} = 0$ , as is easy to see from Cramer's rule, since the odd order determinant  $|c_{ij}|$  is skew-symmetric and therefore identically zero. Hence condition ( $\gamma$ ) is fulfilled.

LEMMA 4. Consider a Pfaffian  $\omega = \sum_{i=1}^{2k-1} X_i dx_i$  satisfying the following conditions:

(A) The  $X$ 's are of class  $C^\mu$ .

(B) None of the  $X$ 's vanish at the origin.

(C) The matrix  $(a_{ij})$  is such that there exists a system of functions,  $\xi_i(x_1, \dots, x_{2k-1})$ , which satisfy the linear equations

$$\sum_{j=1}^{2k-1} a_{ij} \xi_j = 0, \quad \sum_{j=1}^{2k-1} X_j \xi_j = 1.$$

Then there exists a non-singular transformation  $x_i = g_i(y_1, \dots, y_{2k-1})$  of the neighborhood of the origin into itself such that  $\omega$  appears in terms of the  $y$ 's in the form  $\sum_{i=1}^{2k-2} Y_i dy_i + dy_{2k-1}$ , where the  $Y_i$  are functions of  $y_1, \dots, y_{2k-2}$  (independent of  $y_{2k-1}$ ) satisfying the following conditions:

( $\alpha$ ) The  $Y$ 's are of class  $C^{\mu-2}$ .

( $\beta$ ) None of the  $Y$ 's vanish at the origin.

PROOF. Consider the equations

$$(8) \quad \frac{dx_i}{dy_{2k-1}} = \xi_i(x_1, \dots, x_{2k-1}), \quad (i = 1, 2, \dots, 2k-1).$$

It is clear that at least one of the  $\xi$ 's does not vanish at the origin on account of (C). Without loss of generality we assume that  $\xi_{2k-1}(0, \dots, 0) \neq 0$ . It is also clear that the  $\xi$ 's may be taken to be of class  $C^{\mu-1}$ . We can apply Lemma 2 to (8), taking  $f_i = \xi_i$ ,  $n = 2k-1$ ,  $\nu = \mu-1$ . We show that the  $g_i$ , as there defined, satisfy the conditions of the present lemma.

For, on account of (8) and (C), we have

$$\sum_{j=1}^{2k-1} X_j \left( \frac{\partial g_j}{\partial y_{2k-1}} \right) = 1, \quad \sum_{j=1}^{2k-1} a_{ij} \left( \frac{\partial g_j}{\partial y_{2k-1}} \right) = 0.$$

Hence, referring back to (1), we have  $Y_{2k-1} \equiv 1$ , while (2) yields the result that

$$\begin{aligned} \frac{\partial Y_i}{\partial y_{2k-1}} &= \frac{\partial Y_i}{\partial y_{2k-1}} - \frac{\partial Y_{2k-1}}{\partial y_i} = \sum_{j,l=1}^{2k-1} a_{jl} \frac{\partial g_j}{\partial y_i} \frac{\partial g_l}{\partial y_{2k-1}} \\ &= \sum_{j=1}^{2k-1} \left[ \sum_{l=1}^{2k-1} a_{jl} \left( \frac{\partial g_l}{\partial y_{2k-1}} \right) \right] \left( \frac{\partial g_j}{\partial y_i} \right) = 0. \end{aligned}$$

This shows that the  $Y$ 's are all independent of  $y_{2k-1}$ . It remains to prove that conditions ( $\alpha$ ) and ( $\beta$ ) are satisfied.

Condition ( $\alpha$ ) follows from (1) and the fact that, by Lemma

2, the  $g$ 's are of class  $C^{\mu-1}$ . Condition  $(\beta)$  follows from the fact that, by (1) and (4),  $X_i|_0 = Y_i|_0$ ,  $(i = 1, 2, \dots, 2k - 2)$ .

We are now ready to prove our main result.

**THEOREM.** Consider a Pfaffian  $\omega = \sum_{i=1}^{2m} X_i dx_i$  satisfying the following conditions:

- (I) The  $X$ 's are of class  $C^{4m+p}$ ,  $(p \geq 0)$ .
- (II) None of the  $X$ 's vanish at the origin.
- (III) The determinant  $|a_{ij}|$  does not vanish at the origin.

Then there exists a non-singular transformation of coordinates, valid in the vicinity of the origin, which it leaves fixed, such that  $\omega$  appears in the form  $\sum_{i=1}^m y_{2i} dy_{2i-1} + d(\sum_{i=1}^m y_{2i-1})$ . This transformation is of class  $C^{p+1}$ .

**PROOF.** Applying Lemma 3, with  $k = m$  and  $\mu = 4m + p$ , we write  $\omega$  in the form  $e^{x_{2m}} (\sum_{i=1}^{2m-1} X_i dx_i)$ , where the new  $X$ 's are independent of  $x_{2m}$  and in fact satisfy all the hypotheses of Lemma 4, with  $k = m$  and  $\mu = 4m - 2 + p$ . Hence, applying Lemma 4, we can further simplify  $\omega$ , which now appears in the form

$$e^{x_{2m}} \left( \sum_{i=1}^{2m-2} X_i dx_i \right) + e^{x_{2m}} dx_{2m-1}.$$

The new  $X$ 's, by Lemma 4, satisfy all the hypotheses of Lemma 3 (with  $k = m - 1$ ,  $\mu = 4(m - 1) + p$ ), except that we must verify that the new determinant  $|a_{ij}|$  does not vanish. This, however, can easily be done with the help of (3) on account of the fact that the transformation in question is known to be non-singular. Hence we can successively apply Lemmas 3 and 4 again and as a result we have  $\omega$  in the form

$$e^{x_{2m}} e^{x_{2m-2}} \left( \sum_{i=1}^{2m-4} X_i dx_i \right) + e^{x_{2m}} e^{x_{2m-2}} dx_{2m-3} + e^{x_{2m}} dx_{2m-1}.$$

The new  $X$ 's are independent of  $x_{2m-3}, \dots, x_{2m}$  and are of class  $C^{4(m-2)+p}$ . This process may obviously be continued until at the end of the  $m$ th application of Lemmas 3 and 4 we get  $\omega$  in the form

$$e^{x_2} e^{x_4} \dots e^{x_{2m}} dx_1 + \dots + e^{x_{2m-2}} e^{x_{2m}} dx_{2m-3} + e^{x_{2m}} dx_{2m-1}.$$

The transformation at the last application of Lemma 4 is of

class  $C^{p+1}$ . The transformations at the previous steps are of higher class. Hence the compound transformation is of class  $C^{p+1}$ . Finally we make the additional transformation:

$$\begin{aligned}
 y_1 &= x_1, & y_2 &= e^{x_2}e^{x_4} \cdots e^{x_{2m}} - 1, \\
 y_3 &= x_3, & y_4 &= e^{x_4}e^{x_6} \cdots e^{x_{2m}} - 1, \\
 &\dots & & \dots \\
 y_{2m-1} &= x_{2m-1}, & y_{2m} &= e^{x_{2m}} - 1.
 \end{aligned}$$

This transformation, which obviously is analytic with jacobian equal to 1 at the origin, leaves the origin invariant and transforms  $\omega$  into the form  $(y_2+1)dy_1 + \cdots + (y_{2m-2}+1)dy_{2m-3} + (y_{2m}+1)dy_{2m-1}$ . This completes the proof.

It is scarcely necessary to add that, if (I) be replaced by the hypothesis that the  $X$ 's be analytic, the transformation  $T$  furnishing the desired reduction of  $\omega$  will also be analytic. To see this it is only necessary to make slight obvious changes in the previous work, beginning with Lemma 2, where we take the  $f$ 's and consequently the  $g$ 's as analytic.

In the application to the reduction of the Pfaffian equations of dynamics to Hamiltonian form, the condition (II) is of no disadvantage. For the Pfaffian differential equations are unaltered if to each  $X_i$  a constant is added. Assuming then that (II) holds (for each value of  $t$ ) the equations may be written in variational form as follows:

$$\delta \int \left[ \sum_{i=1}^m y_{2i} \left( \frac{dy_{2i-1}}{dt} \right) + H \right] dt = 0,$$

where now, since the  $X$ 's (and consequently  $T$ ) depend in general periodically upon  $t$ ,  $H$  is not necessarily equal to  $Q$ , but rather, of course,

$$H = Q + \sum_{i=1}^{2m} X_i \left( \frac{dx_i}{dt} \right).$$