ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THIS SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

192. Dr. W. I. Miller: *Fundamental regions for the simple group of order 168 in* S_4.

Fundamental regions have been determined for certain groups in S_4 (Price, American Journal of Mathematics, vol. 40 (1918), pp. 108–112). In this paper fundamental regions for the ternary G_{168} are obtained by the use of 21 forms of degree four in which every term is of degree two in the variables x_1, x_2, x_3, and of degree two in the conjugate imaginary variables. These forms are written as the differences of seven positive definite forms, so that their behavior under the group may be studied by means of the permutation group of degree seven. If we set $x_1/x_3=x+iu$, $x_2/x_3=y+iv$, each of these forms, equated to zero, represents a hypersurface in S_4. These hypersurfaces divide the S_4 into 7! regions, each of which has 168 conjugates under the group. A fundamental region is obtained by selecting one region from each of the 30 sets of conjugate regions. Points on the hypersurfaces are also considered. (Received May 15, 1933.)

193. Professor H. S. Vandiver: *On algebraic rings and Abelian groups.*

The methods employed in this article depend on certain representations of the elements of a finite Abelian group (in particular what is called the p-adic representation) which lead also to two possibly new proofs of the existence of a basis for this type of group. Very simple derivations of the known main results and some new results are obtained in the theories of finite fields and rings. (Received May 19, 1933.)

194. Professor L. E. Dickson: *Every large number is a sum of nine values of a cubic polynomial in* x.

The author gives cases of universal theorems described by the title (Received May 4, 1933.)

195. Professor B. A. Bernstein: *A set of four postulates for Boolean algebra in terms of the "implicative" operation.*

The author obtains, as the main object of his paper, a set of four independent postulates for Boolean algebra expressed in terms of the operation

"$\supset$," the "relation of implication" of *Principia Mathematica*. The number of postulates is thus considerably smaller than the number (eight, including an inadvertently omitted existence postulate) in Huntington's set of postulates expressed in terms of "$\supset$" (Proceedings of the National Academy of Sciences, vol. 18(1932), p. 180). The author also shows that there is a very close relation between the operation "$\supset$" and the operation "$-$" of "exception," by virtue of which relation any set of postulates in terms of "$-$" is essentially also a set of postulates in terms of "$\supset$," and vice versa. Finally, the author derives from his postulates the "theory of deduction" of *Principia Mathematica*. This derivation brings out the fact, perhaps more directly than has been done before, that the propositions of the theory of deduction are all propositions in the general logic of classes. (Received May 16, 1933.)

196. Mr. A. M. Tuttle: *A system of independent mathematical postulates for Dirac's quantum mechanics.*

This paper exhibits a system of postulates which are sufficient to furnish a rigorous formal foundation for a type of mathematics used by Dirac in his quantum mechanics (Dirac, *Principles of Quantum Mechanics*, Oxford, 1930). The first twelve postulates furnish a calculus for the system. The notion of a general linear process has been introduced instead of the processes of summation and integration which Dirac uses. In the first twelve postulates this process is not so restricted as to exclude an ordinary integration and many of Dirac's results are obtained upon this basis. However, in order to derive all of Dirac's expansion theorems, it has been necessary to add two more postulates which so restrict the general linear process as to exclude an ordinary integration. The postulates are shown to be consistent and independent. (Received May 19, 1933.)

197. Professor H. S. Vandiver: *Summary of results and proofs on Fermat's last theoem.* Seventh paper.

The main result obtained is as follows. If l is an odd prime and $x^l+y^l+z^l=0$ is satisfied in rational integers, none zero, and prime to the odd prime l, then the second factor of the class number of the algebraic field defined by $e^{2i\pi/l}$ is divisible by l. Other criteria are also found and the proofs indicated. (Received May 19, 1933.)

198. Professor A. A. Albert: *Integral domains of rational generalized quaternion algebras.*

In recent years several papers on the integers of generalized quaternion algebras over R have appeared and various special cases have been considered. But the general case has not yet been completed; in fact the question as to whether it would not be possible to eliminate many of these special cases has not been discussed. The present paper utilizes simple algebraic transformations as well as the theory of ternary quadratic null forms to obtain a remarkable canonical form for any generalized quaternion algebra over R. Moreover the author shows that for algebras in the canonical form there is a single domain of integrity containing the basal units, and determines this domain. (Received May 20, 1933.)

199. Professor H. L. Olson: *Bilinear correspondences in two and in three dimensions.*

Every bilinear form in $n+n$ variables $x_1, \cdots, x_n, y_1, \cdots, y_n$ can be interpreted geometrically as determining a correspondence, in terms of homogeneous coordinates, between points and hyper-planes (linear subspaces of $n-2$ dimensions) in a flat space of $n-1$ dimensions. This paper classifies, projectively, such correspondences in $3+3$ and $4+4$ variables, and presents a canonical form for each class. (Received May 17, 1933.)

200. Miss A. V. Newton: *Consecutive covariant configurations at a point of a space curve.*

The purpose of this paper is to make some contributions to the projective differential geometry of space curves. There are many configurations covariantly connected with a point P of such a curve, such as tangent line, osculating cone, and Halphen point. As the point P moves along the curve each covariant point describes a curve, each covariant line describes a surface, and each covariant surface has an envelope. On considering a point Q infinitesimally near to the point P and setting up a transformation of coordinates between local coordinate systems at P and Q, it becomes possible to obtain considerable information about these curves and envelopes. In particular, one can find the tangent lines of the curves, the characteristics of the envelopes, and the foci of the edges of regression of the envelopes. (Received May 20, 1933.)

201. Professor A. H. Copeland: *Consistency of the conditions determining Kollektivs.*

A rigorous basis for the statistical theory of probability is obtained by the establishment of the consistency of the conditions determining the set of *Kollektivs*. The author shows that, in a certain sense, the physical situation must be in agreement with this theory. In this respect the theory differs from all other physical theories. Moreover, the *Kollektiv* establishes a closer relationship between physical situations and their mathematical formulations. The conditions determining *Kollektivs* are due to von Mises. Consistency can be established by means of the construction of sequences of points satisfying the conditions. The consistency of the conditions for *Kollektivs* of a very special type can be shown by means of the *nombres normaux* of Borel. For this type the conditions assume a much simpler form. In this paper it is shown that a slight modification of the conditions for the more general type is essential to achieve consistency. It is further shown by a method similar to that of Borel that these modified conditions can be satisfied. (Received May 17, 1933.)

202. Mr. T. N. E. Greville: *Invariance of the property of admissibility under certain general types of transformations.*

This paper is concerned with a set of transformations on certain sequences, which include the type of *Kollektiv* called by von Mises "*die einfachste alternative*" and the admissible number of Copeland. It is possible to formulate completely in terms of the transformations of this set a large class of problems arising in the classical theory of probability for which the operations given by von Mises are not adequate. The consistency of the assumptions of the classical theory is investigated by means of certain properties of the sequences which are invariant under these transformations. The method used in demonstrating the invariance of the properties in question is similar to that employed by Borel in proving the existence of *nombres normaux*. (Received May 17, 1933.)

203. Dr. Francis Regan: *The application of the theory of admissible numbers to time series with variable probability.*

A time series is a sequence of occurrences which may be represented by a set of points on a time axis. In this paper the author is interested in the time series whose points are determined with relation to a distribution function which assigns a definite probability to every interval of the time axis. The probability varies according to the length and position of the interval. The concept of admissibility is extended to this type of time series by setting up, in accordance with the above distribution function, an imaginary set of points on the time axis. The statistical point of view of probability is applied to this series and the set is required to satisfy the fundamental assumptions of the theory of probability. Thus the sequences of successes and failures must be represented by the digits of admissible numbers. For every interval a different set of conditions is obtained and hence the number of conditions imposed has the power of the continuum. The author shows how to obtain a set of points for which these conditions are satisfied. (Received May 17, 1933.)

204. Professor H. M. Gehman: *Points of connectivity.*

A point of connectivity of a set M is a point p of M such that every connected subset of M containing p is a non-cut set of M. It is shown that there are exactly ten types of sets which have points of connectivity. The proofs are based on the set of three axioms mentioned in a previous abstract (No. 39-5-152). The results of this paper generalize theorems due to Zarankiewicz (Fundamenta Mathematicae, vol. 5 (1924), pp. 11-13). (Received May 19, 1933.)

205. Dr. N. E. Rutt: *Some theorems on triodic continua.*

Concerning any compact plane continuum K, it is proved in this paper that if K does not divide the plane and is, for any point k of it, expressible as the sum of two of its subcontinua neither of which conains K and both of which contain k, then K is triodic; and that if K, for any point k of it, is the sum of three of its subcontinua each of which contains k and none of which is contained in the sum of the other two, then K contains a triod. Concerning any collection of non-compact plane continua it is proved that if each is the sum of two of its non-compact proper subcontinua, then the elements of the collection are not both mutually exclusive and uncountable. (Received May 16, 1933.)

206. Mr. C. H. Wheeler, III: *A type of homogeneity for continuous curves.*

The author investigates in this paper conditions under which a compact, locally connected continuum M may be cyclic element homogeneous, i.e., given any two non-degenerate cyclic elements of M, there exists a homeomorphism which sends M into itself in such a way that one of these elements is sent into the other. The case where there are only a finite number of non-degenerate cyclic elements is completely treated. In addition, the investigation has been extended to some cases where the number of such elements is infinite. (Received May 18, 1933.)

207. Mr. R. R. Lyle: *Centers and axes of symmetry.*

The idea of a set of points in the euclidean plane being symmetrical to a center or to an axis is not new. As far as the author knows, no one has previously studied the relations between centers and axes of symmetry for sets having more than one center and more than two axes. This paper contains numerous theorems concerning relations between the number of centers and axes of symmetry of a given set. Examples are given showing the existence of sets having the properties stated in the theorems. (Received May 19, 1933.)

208. Mr. B. B. Sharpe: *Arcs of symmetry.*

We shall say that the arc AB is an arc of symmetry of a set M if $M-(M\cdot AB)$ can be expressed as the sum of two mutually separated sets L and R, such that there exists a homeomorphism T having the following properties: (1) $T(M+AB)=M+AB$; (2) if $X=AB$, then $T(X)=X$; and (3) $T(L)=R$. It is shown that an arc of symmetry is an invariant of the homeomorphic geometry of any space, and that if a homeomorphism T exists, then there is a homeomorphism U having properties (1), (2), and (3) and also such that $U^2(X)=X$ for each point of $M+AB$. Properties of sets which have an arc of symmetry and also sufficient conditions for a set to have an arc of symmetry are considered. The following theorem is proved: If M can be expressed as the sum of a finite number of simple closed curves $C_1, C_2, \cdots, C_n$, each pair of which have a set K in common such that for $i, j=1, 2, 3, \cdots, n$, the points of K occur on C_i in the same cyclic order as they occur on C_j, then M has an uncountably infinite number of arcs of symmetry. (Received May 19, 1933.)

209. Professor J. W. Campbell: *The clock problem in relativity.* Second paper.

The writer's first paper on this subject appeared in the January, 1933, number of the Philosophical Magazine. The device of using a fictitious gravitational potential was there used, and the solution obtained is applicable to paths of any shape and described in any manner, provided only that the speed is not too great. The present paper makes use of rotating axes, and the solution is applicable to motion on a straight line which is not in the vicinity of ponderable matter, though without any restriction as to the magnitude of the speed. Moreover, for the case treated, there is a more vivid picture obtained of the phenomenon of time distortion than is obtained by using a fictitious gravitational potential. (Received May 15, 1933.)

210. Professor Frederick Creedy: *A hyperbolic vector diagram and symbolic method.*

The paper relates to the application of a two dimensional hyperbolic geometry to the study of electrical transients of the form $Ae^{at}+Be^{bt}$. It introduces a geometrical operator k which changes a vector whose components are (a, b) into another whose components are (b, a). Thus $k(a, b)=(b, a)$ and $k^2=1$. Extranormal numbers of the form $x+ky$ are defined and an extranormal number plane corresponding to the complex number plane (but with the relations of hyperbolic instead of euclidean geometry) studied. In terms of this system a transient may be completely represented by Ae^{pt} (A and p being extranormals). Geometrical constructions derived from hyperbolic geometry are developed for the study of various problems relating to electrical transients. The scalar product of two vectors $a+kb$ and $c+kd$ is given a double definition in the plane, one serving for euclidean and the other for hyperbolic geometry, so that the same formulas are capable of two geometrical interpretations, one in each system. (Received May 19, 1933.)

211. Dr. P. G. Hoel: *Certain problems in the theory of closest approximation.*

This paper is concerned with properties of minimizing sums in the field of approximation theory of least mth powers for values of m less than one. With suitable restrictions it is shown that the orthogonality property of the error function holds; and, with the introduction of root multiplicity bounds as a generalization of root multiplicities, that certain continuous error functions possess at least a fixed total number of roots and root multiplicity bounds. Under rather light restrictions it is shown that the coefficients of a minimizing sum are continuous functions of m for all positive values of m. (Received May 18, 1933.)

212. Dr. W. T. Reid: *Discontinuous solutions of the non-parametric problem of Mayer in the calculus of variations.*

The question of discontinuous solutions in the simple problem of the calculus of variations in parametric form has been treated rather thoroughly (See L. M. Graves, this Bulletin, vol. 36 (1930), pp. 831–846), whereas for the non-parametric problem relatively few results have been obtained. This paper treats discontinuous solutions for the general problem of Mayer in non-parametric form, and in terms of the characteristic numbers of a boundary value problem associated with the second variation there is given an effective extension of the Jacobi condition. Sufficient conditions for a minimum are then established by the method used in the case of extremal arcs by Bliss and Hestenes (Transactions of this Society, vol. 35 (1933), pp. 305–326 and pp. 479–490). For the accessory boundary value problem in terms of which the analogue of Jacobi's condition is phrased the boundary conditions apply at more than two points, but it is significant that this problem may be transformed into one for which the boundary conditions apply at only two points, and of the type recently considered by the author (American Journal of Mathematics, vol. 54 (1932), pp. 769–790). (Received May 19, 1933.)

213. Dr. M. R. Hestenes: *Sufficient conditions for the problem of Bolza in the calculus of variations.*

Sufficient conditions for the problem of Bolza have been given by Morse (American Journal of Mathematics, vol. 53 (1931), pp. 517–546) and Bliss (Annals of Mathematics, vol. 33 (1932), pp. 261–274) for arcs which are not only normal relative to the end conditions but also normal on every sub-interval. The latter normality condition prevents these sufficiency conditions from being applicable without further modification to the general problem of Mayer and to the problems of the various classes of Caratheodory (Commentarii Mathematici Helvetici, vol. 5 (1933), pp. 1–19). In the present paper a set of sufficient conditions for the problem of Bolza is given assuming only normality relative to the end conditions. No assumptions regarding normality on the sub-intervals are needed. The conditions here given are therefore applicable at once to the problems of Mayer and of Caratheodory. This result is obtained by replacing the usual Mayer condition by a new condition involving a bilinear form with certain linear conditions adjoined. This new condition is a consequence of the theory of broken extremals applied to the problem of the second variation. The results of this paper are extensions of those recently announced by the author for the problem of Lagrange (Abstract No. 39-5-124). (Received May 23, 1933.)

214. Mr. H. L. Turrittin: *Asymptotic solutions of certain ordinary differential equations associated with multiple roots of the characteristic equation.* Preliminary report.

In certain instances the asymptotic nature of a fundamental set of solutions of an ordinary linear differential equation containing a parameter ρ has been ascertained in the neighborhood of $\rho = \infty$. The present paper contains the first results obtained by the author in attempting to extend Birkhoff's analysis (Transactions of this Society, vol. 9 (1908), pp. 219–231) to include certain cases in which the characteristic equation possesses multiple roots; subject to the proviso that if any two roots are equal, they are identically equal throughout the entire interval under consideration. The asymptotic nature of a fundamental set of solutions is deduced subject to the assumption that the actual determination of the formal solutions of the type described by Tamarkin (Master's Thesis, Petrograd, 1917) involves at most the solution of first, and not higher, order differential equations. (Received May 19, 1933.)

215. Professor I. A. Barnett: *Functional invariants of integro-differential equations.*

In a previous paper the author pointed out how the study of the invariants and covariants of a system of linear differential equations of the second order could be extended to the analogous consideration of the functional invariants and covariants of an integro-differential equation of the second order. In this paper an existence theorem concerning the solutions of such equations is considered. It is shown further that there is an infinite system of functional invariants involving derivatives of all orders. These invariants turn out to be the traces of certain kernels arising from the coefficients of the original equation. Finally, the author suggests an application of these results to the study of projective properties of surfaces in a certain type of function space. (Received May 19, 1933.)

216. Dr. E. W. Titt (National Research Fellow): *Cauchy's problem for systems of second order partial differential equations.*

By a method similar to that used by E. Cartan (Bulletin de la Société Mathématique de France, vol. 59 (1931)) on systems of first order equations, the author obtains a canonical form for systems of second order partial differential equations, linear in the second derivatives. New considerations enter into the second order problem; for example, in the first order case there is only one possibility for a canonical form while in the second order case it is not evident which of two possibilities will give the desired results. Use of Riquier's theory of orthonomic systems leads to a clear statement of Cauchy's problem for the systems under consideration. The characteristic surfaces are those over which the results on Cauchy's problem fail to apply. The theory applies to Einstein's gravitational equations for free space. A theorem which shows that Cauchy's problem for a characteristic fails by becoming indeterminate is proved using the theory of normal systems of differential equations of Thomas and Titt (Annals of Mathematics, ser. 2, vol. 34 (1932)), an extension of Riquier's work. For arbitrary coordinate systems the present theory handles the second order invariantive systems treated by Thomas without using the theory of differential invariants of generalized spaces and without reducing the system to an equivalent system of first order equations. (Received May 22, 1933.)

217. Mr. H. H. Alden: *Properties of differential equations arising from properties of the solution curves.*

F. R. Bamforth and the author have shown (this Bulletin, Abstract No. 39-1-7) that $dy/dx=f(x, y)$ may have a unique solution extending from boundary to boundary through each point of the region of definition of $f(x, y)$ without $f(x, y)$ being continuous or satisfying any of a number of well known uniqueness conditions. In the present note it is shown that the hypotheses usually placed on $f(x, y)$ to insure that the solutions have such further properties as differentiability with respect to initial conditions and analyticity are actually necessary for the solutions to have these properties. (Received, May 13, 1933.)

218. Professor O. D. Kellogg: *Converses of Gauss' theorem on the arithmetic mean.*

Let T be a bounded domain of the plane. Let the function $u(p)$ be continuous in T. Let $c(p)$ denote a circle about p as center, and let $A(u, c)$ denote the arithmetic mean of u on c. Then, according to Koebe's converse of Gauss' theorem, u is harmonic in T if, for each point p in T, $u(p)=A\{u, c(p)\}$ for every circle $c(p)$ contained in T. The question arises as to the properties of u if one supposes only that, for each p of T, $u(p)=A\{u, c(p)\}$ for some one circle contained in T, or, more generally, if one supposes only that $u(p)$ is equal, at each p of T, to the solution of the Dirichlet problem, with boundary values u, for some normal domain containing p and strictly contained in T. In this paper results are obtained in regard to a function u of this character. These results are concerned chiefly with the bounds of u, and conditions on u in the neighborhood of the boundary of T that imply that u is harmonic in T. The reasoning is valid in space, and holds, under suitable restrictions on u, even when the boundaries of the normal domains have points in common with the boundary of T. The results were obtained by the late Professor Kellogg. They were prepared for publication by J. J. Gergen. (Received May 13, 1933.)

219. Dr. Hassler Whitney (National Research Fellow): *Derivatives, difference quotients, and Taylor's formula.*

Let $f(x)$ be measurable in the closed interval I. A necessary and sufficient condition that $f(x)$ be a polynomial of degree at most $m-1$ is that the mth difference quotient $\Delta_h^m f(x)=\sum_{i=0}^m(-1)^{m-i}\binom{m}{i}f(x+ih)/h^m\to 0$ uniformly as $h\to 0$. A necessary and sufficient condition that $f(x)$ have a continuous mth derivative is that there exist a continuous function $f_m(x)$ in I such that $\Delta_h^m f(x)\to f_m(x)$ uniformly as $h\to 0$, or, that there exist functions $f_0(x)=f(x)$, $f_1(x),\cdots,f_m(x)$ in I such that $f(x+h)=\sum_{i=0}^m f_i(x)h^i/i!+R(x, h)$, where $R(x, h)/h^m\to 0$ uniformly as $h\to 0$, then $d^if(x)/dx^i=f_i(x)$, $(i=1,\cdots,m)$. (Received May 13, 1933.)

220. Professor J. F. Ritt and Dr. J. L. Doob (National Research Fellow): *Systems of algebraic difference equations.*

A definition is given of *irreducible* systems of algebraic difference equations. It is shown that every system of such equations is equivalent to a finite set of irreducible systems. (Received May 17, 1933.)

221. Dr. W. J. Trjitzinsky: *The general case of non-homogeneous linear differential equations.*

In the present paper the author considers the asymptotic nature of solutions, in the complex plane and in the vicinity of a singular point, of equations of the type indicated in the title. The treatment is given for the general case of *unrestricted* roots of the corresponding characteristic equations. This treatment is based on another work of the author in which the analytic theory of the analogous problem for homogeneous differential equations is developed. (Received May 20, 1933.)

222. Professor J. A. Shohat and Dr. Clement Winston: *On mechanical quadratures.*

This paper is an extension of one on the same subject presented to the Society at its April, 1933, meeting in New York by Dr. Winston. It gives a further discussion of the coefficients $H_{i,n}$ in the mechanical quadratures formula $\int_a^b f(x)p(x)dx=\sum_{i=1}^n H_{i,n}f(x_{i,n})+R_n(f)$, $(H_{i,n}=\int_a^b(\phi_n(x)p(x)dx)/((x-x_{i,n})\cdot\phi_n'(x_{i,n}))$, $\int_a^b\phi_n(x)\phi_m(x)p(x)dx=\delta_{m,n})$ for the classical orthogonal polynomials, and especially for those of Hermite. Upper and lower bounds for the coefficients $H_{i,n}$ are obtained using their extremal properties and certain expressions involving $K_n(x) \equiv \sum^{n}_{=1}\phi_i^2(x)$; also a differential equation of the first order for $K_n(x)$ is derived. Special considerations in the Hermite case lead to some interesting asymptotic relations, as, for example, $\lim_{n\to\infty} K_n(x)/(2n)^{1/2} = e^{x^2}/\pi$, $(|x| < (2n - cn^{1/3})^{1/2}$, c constant). (Received May 20, 1933.)

223. Dr. M. R. Hestenes: *A note on the Jacobi condition for parametric problems in the calculus of variations.*

The present note gives a simple modification of the treatment of the Jacobi condition given by Bliss (Transactions of this Society, vol. 17 (1916), pp. 195–206). Instead of defining conjugate points in terms of solutions η_i of the Jacobi equations which satisfy the conditions $y_i'\eta_i = 0$, as was done by Bliss, they are defined in terms of solutions η_i which satisfy the conditions $y_i'\eta_i' =$ constant. With the help of these solutions the usual criteria for conjugate points are deduced. Since there are but $2n-2$ linearly independent solutions of the first type and $2n$ of the second type, the treatment here given seems to be simpler and more symmetric than that of Bliss and is almost identical with that usually given for the non-parametric problems. (Received May 23, 1933.)

224. Professor D. N. Lehmer: *Franklin magic squares.*

In a letter to Peter Collinson, Benjamin Franklin describes a method of constructing magic squares of even order, illustrating it with two squares, one of order 8 and one of order 16. These squares are magic, not only in the rows and columns and "broken diagonals" but also *im kleinen*, that is to say; the sum of the elements in any sub-square of order 2 is constant over the whole square. In this paper the author develops what is without doubt the method employed by Franklin in constructing these squares and shows how it may be employed in squares of all even orders, and brings the whole into a unified theory. (Received May 10, 1933.)

225. Professor A. A. Albert: *A generalization of a theorem of Dirichlet.*

In 1840 Dirichlet outlined a proof of the theorem that every properly primitive binary with non-square discriminant represents infinitely many primes. This theorem was later proved by H. Weber, A. Meyer, and F. Mertens, but seems never to have been extended to n-ary or to non-classical forms. In the present paper the author applies the Hilbert Irreducibility Theorem to show that every properly primitive (classical or non-classical) n-ary quadratic form is equivalent to a form $f(x_1, \cdots, x_n)$ whose sub-forms $f_r = f(x_1, \cdots, x_r, 0, \cdots, 0)$ are all properly primitive and non-singular while f_2 has non-square discriminant. The extension of the Dirichlet theorem to non-classical forms is immediate and proves that every n-ary in the above canonical form represents infinitely many primes, in particular those represented by f_2. (Received May 20, 1933.)

226. Professor C. G. Latimer: *Note on the invariants of the class group of a cyclic field.*

Let F be an algebraic field which is cyclic and of odd degree e, with respect to the rational field. The classes of ideals in F are the elements of an Abelian group G, which is characterized by certain prime power invariants. Let the distinct fields defined by the various e-th roots of unity $\neq 1$ be $K_1, K_2, \cdots, K_n$. The purpose of this note is to prove the following theorem. Let p be a rational prime and k a positive integer. If the prime ideal divisors of p in $K_1, K_2, \cdots, K_n$ are of degree $g_1, g_2, \cdots, g_n$ respectively, the number of invariants of G which are equal to p^k may be written in the form $g_1x_1+g_2x_2+\cdots+g_nx_n$ where each $x \geqq 0$. Since the class number h, of F, is the product of the invariants, it follows that h is the product of the norms of n ideals, one in each of the fields K_i. For the case where e is a prime, this result was previously obtained by the writer in a different manner. (Transactions of this Society, vol. 35 (1933), p. 417.) (Received May 19, 1933.)

227. Dr. H. P. Thielman: *Note on Riemannian function space geometry.*

In this note the following result is obtained: *Every Riemannian function space can be immersed in a euclidean function space.* A Riemannian function space is here defined in accordance with the definition found in the literature (A. D. Michal, American Journal of Mathematics, vol. 50, p. 516) as one in which the element of length is given by $(ds)^2=\int_a^b\int_a^b g(\alpha, \beta)\partial y(\alpha)\partial y(\beta)d\alpha d\beta+\int_a^b g(\alpha)\partial^2 y(\alpha)d\alpha>0$ where $g(\alpha, \beta)=g(\beta, \alpha)$, $g(\alpha)\neq 0$ in (a, b), and $\partial y(\alpha)$ is continuous and does not vanish identically in (a, b). If $g(\alpha, \beta)\equiv 0$, $g(\alpha)\equiv 1$ the space is a euclidean function space. The equivalence is established under the group of linear functional transformations of the third kind: $g(s)=K(s)\bar{y}(s)+\int_a^b K(s, \alpha)\bar{y}(\alpha)d\alpha$, $K(s)\neq 0$ in (a, b). (Received May 23, 1933.)

228. Dr. A. E. Ross (National Research Fellow): *Positive quaternary quadratic forms representing all but a finite number of integers.*

Ramanujan, having determined all universal positive quaternary quadratic forms of the form $ax^2+by^2+cz^2+du^2$, proposed the problem of determination of all such forms representing all but a finite number of integers (Collected Papers, Cambridge University Press (1927), p. 169). This problem was later solved by H. P. Kloosterman (Acta Mathematica, vol. 49 (1926), pp. 407–64). In the present paper, the author proposes to study positive quaternary quadratic forms representing all but a finite number of integers, without imposing the restriction that these forms should contain no cross-product terms. (Received May 23, 1933.)

229. Dr. B. F. Kimball: *The generalized Bernoulli polynomial and its relation to the Riemann zeta function.*

The generalized Bernoulli polynomial $B_s(x)$ is defined as the solution of the difference equation $B_s(x+1)-B_s(x)=sx^{s-1}$, x real and positive, s a complex variable; this solution to be analytic in s and such that $\lim_{x\to\infty}B_s(x)=0$, $R(s)<0$. It is found that under the above conditions $B_s(x)$ is an entire function of s, that it is unique, and that it reduces to the classical Bernoulli polynomials when s is a positive integer or zero. It is found that the Riemann zeta function is related to it by the equation $(-s)\zeta(1-s, x)=B_s(x)$ for all values of s, x real and positive. Thus $-\zeta(-s, x)$ is the unique solution of the difference equation $f(s, x+1)-f(s, x)=x^s$, $s\neq -1$, x real and positive, which is analytic in s except at $s=-1$, and such that $\lim_{x\to\infty}f(s, x)=0$, $R(s)<-1$. (Received June 5, 1933.)

230. Dr. D. C. Duncan: *The completely symmetric rational self-dual septimic.*

A discussion of the completely symmetric elliptic self-dual septimic will appear shortly in this Bulletin. The present paper treats the completely symmetric rational septimic, having five of each of the simple singularities, cusps, inflexions, crunodes, acnodes isolated and ordinary double-tangents. The ten collineations and ten correlations under which the curve is invariant are listed. Of the correlations five are polarities by rectangular hyperbolas and one a polarity by an imaginary circle touching the hyperbolas at the ends of their conjugate axes. (Received June 5, 1933.)

231. Dr. R. E. A. C. Paley: *A special integral function.*

In this paper the late R. E. A. C. Paley gives an example of an entire function which for the minimum modulus $m(r)$ is bounded, but which possesses no asymptotic path. The "segments" of the series defining the non-differentiable Weierstrassian function play essential roles in this construction. (Received June 5, 1933.)

232. Mr. T. P. Palmer: *Postulates for logic.*

The following set of independent postulates suggested by papers of H. M. Sheffer, N. G. P. Nicod, and C. E. Van Horn, is proved equivalent to E. V. Huntington's postulates for the *informal Principia*. The base is a class, T, a class, F, and an operation, $|$, (stroke). 1. If p in T and q in T then $p|q$ in F. 2. If p in T and q in F then $p|q$ in T. 3. If p in F and q in T then $p|q$ in T. 4. If p in F and q in F then $p|q$ in T. 5. If p in T then p not in F. It is shown that if, in the *formal Principia*, any one proposition is both true and false, then every proposition is both true and false and the *formal Principia* becomes trivial. Thus the addition of postulate 5 above to *formal Principia* is not abhorrent. The postulates (except 5) are proved from *formal Principia*, and, with K suitably defined, *formal Principia* is proved from these. The above postulates are generalized to give a system with multiple truth values, K defined, and *Principia's* postulates are proved for this system. (Received June 8, 1933.)

233. Professor G. A. Miller: *Groups in which every operator has at most a prime number of conjugates.*

The author proves that if every non-invariant operator of a group has a given prime number of conjugates under this group then the order of the commutator subgroup of this group is this prime number and its operators are invariant under the group. It is also proved that if a group contains more than one Sylow subgroup of order p^m then it contains a set of conjugate operators whose order is a power of p and whose number exceeds p. (Received June 14, 1933.)

234. Professor A. J. Maria: *Concerning the equilibrium point of Green's function for an annulus.*

Let r and r_0 be the distance from the center of the two concentric circles of the equilibrium point and pole, respectively. The points lie on the same diameter. It is shown that $dr/dr_0 > 0$ for any position of the pole interior to the ring on a fixed diameter, and that lim $dr/dr_0 = 0$ as r_0 approaches the inner or outer boundaries of the ring. A formula is given for the maximum displacement of r from the geometric mean circle. (Received June 15, 1933.)

235. Professor E. V. Huntington: *A second set of independent postulates for the informal Principia Mathematica.*

The set given in Appendix II of the author's paper in the Transactions of this Society, vol. 35 (1933), pp. 274–304, 557–558, contains a redundancy. Example 5 (page 304) is erroneous, and Postulate 5 (page 301) is deducible from Postulates 1, 2, 3, 4, 6, 7. A second set of postulates for the same system is the following. *Base*: $(K, T, +, ')$, where T is a proper subclass in K, that is, K is dichotomized into two non-empty, non-overlapping subclasses T and F. Postulates: 1°. If a and b are in K, then $a+b$ is in K. 2°. If a is in K, then a' is in K. 3°. If b is in T, then $a+b$ is in T. 4°. If b is in T, then $b+a$ is in T. 5°. If a is in F, then a' is in T. 6°. If a is in T, then a' is in F. 7°. If $a+b$ is in T and a is in F, then b is in T. In this set no postulate involves more than two elements. A similar set is obtained by interchanging $+$ and $\times$, and T and F. (Received June 16, 1933.)

236. Professor I. J. Schwatt: *The sum of like powers of a series of numbers forming an arithmetical progression and the Bernoulli numbers.*

Since the beginning of the eighteenth century mathematicians, in expanding $\sum_{k=1}^{n-1}(\pm 1)^{k-1}k^p$ into an explicit function in powers of n, have always assumed the coefficients of the expansion to contain a Bernoulli number in order, that is, the Bernoulli numbers are assumed to be known. The author has succeeded in devising methods which render a direct expansion of the function and in this way numerical expressions for the Bernoulli numbers. By means of this method the general term of the expansion, and therefore the general Bernoulli number, can be found. This was impossible by the method used heretofore. (Received June 17, 1933.)

237. Dr. J. W. Bower: *The problem of Lagrange with finite side conditions.*

In this paper two methods are used for the development of a complete theory of solution of the problem of Lagrange with finite side conditions.

The first transforms it by differentiation of the side conditions and reduction of the end conditions into a problem of Lagrange with differential equations as side conditions and first end-point variable. Admissible arcs for the new problem are normal relative to the end conditions and possess normality on every subinterval relative to similarly reduced end conditions but not with respect to fixed end-points as is required in the theories hitherto given for problems with variable end-points. The second method replaces the original problem by an equivalent simple problem of the calculus of variations in xz-space. This is done by adjunction of suitably chosen auxiliary equations to the side conditions and solution of the resulting system for the variables y in terms of new variables z. The well-known necessary and sufficient conditions for a minimum for this simple problem are interpreted for the original problem in xy-space, and the auxiliary functions are eliminated from the resulting conditions. Identical theories of solution of the original problem are deduced by application of these two methods. (Received June 19, 1933.)

238. Mr. E. N. Oberg: *The approximate solution of integral equations.*

The aim of this paper is to investigate the approximate representation of the solution of a given Fredholm linear integral equation $L(u)=f(x)$ by means of a polynomial. The criterion of approximation is the minimizing of the integral of $|f(x)-L(P_n)|^m$, where $P_n(x)$ is a polynomial of the nth degree, and the principal problem is that of the convergence of $P_n(x)$ toward $u(x)$ as n becomes infinite, $u(x)$ being the unique continuous solution of the integral equation. This problem for $m=2$ has been discussed recently by Picone, Krawtchouk, Enskog, and others, but the methods and results are different. The convergence proof for $m\geqq 1$ is based indirectly on Hölder's inequality, and for $m<1$ on an extension of Markoff's theorem on the derivative of a polynomial. (Received June 17, 1933.)