ON THE DERIVATION OF NECESSARY CONDITIONS FOR THE PROBLEM OF BOLZA*

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1. *Introduction.* The problem of Bolza with variable end-points in the calculus of variations† is that of finding in a class of arcs

$$(1)\qquad y_i = y_i(x), \quad (i = 1, 2, \cdots, n;\ x_1 \leqq x \leqq x_2),$$

satisfying the differential equations and end-conditions

$$(2)\qquad \phi_\alpha(x, y, y') = 0, \qquad (\alpha = 1, 2, \cdots, m < n),$$

$$(3)\qquad \psi_\mu[x_1, y(x_1), x_2, y(x_2)] = 0, \quad (\mu = 1, 2, \cdots, p \leqq 2n + 2),$$

one which minimizes an expression of the form

$$(4)\qquad I = G[x_1, y(x_1), x_2, y(x_2)] + \int_{x_1}^{x_2} f(x, y, y')dx.$$

Morse and Myers‡ have recently studied this problem in a somewhat different form. They seek to find in the class of arcs (1) satisfying the differential equations (2) and the end-conditions

$$(5)\qquad x_s = x_s(\alpha_1, \cdots, \alpha_r),\ y_{is} = y_{is}(\alpha_1, \cdots, \alpha_r), \qquad (s = 1, 2;\ i = 1, 2, \cdots, n),$$

one which minimizes a sum

$$(6)\qquad I = \theta(\alpha_1, \cdots, \alpha_r) + \int_{x_1}^{x_2} f(x, y, y')dx.$$

It has been customary to assume that the matrix of derivatives of the functions ψ_μ in (3) with respect to the variables x_s,

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† Bolza, *Über den "Anormalen Fall" beim Lagrangeschen und Mayerschen Problem mit gemischten Bedingungen und variablen Endpunkten*, Mathematische Annalen, vol. 74 (1913), pp. 430–446. Bliss, *The problem of Bolza in the calculus of variations*, Annals of Mathematics, (2), vol. 33 (1932), pp. 261–274. In the text here the latter paper will be referred to by the symbol B_2.

‡ *The problems of Lagrange and Mayer with variable end-points*, Proceedings of the American Academy of Arts and Sciences, vol. 66 (1931), pp. 235–253. In the text this paper will be designated by the symbol *MM*.

$y_{is} = y_i(x_s)$, $(s = 1, 2; i = 1, \cdots, n)$, is of rank p at the end-values of the particular minimizing arc E_{12} to be studied,* an assumption which insures the non-singularity of the end-values and the independence of the conditions (3). For the parametric form (5) Morse and Myers do not assume that the matrix of derivatives of the functions (5) is of rank r at the parameter values defining the ends of E_{12},† and hence the problem as formulated by them seems to have greater generality. In a footnote on page 244 of their paper, Morse and Myers call attention to this fact and make the statement that the necessary conditions for a minimum deduced by them from the first variation can not easily be derived from the results given by Bliss for the corresponding Lagrange problem.‡

In the following pages the authors wish to show how the methods used by Bliss for the Lagrange problem can be adapted with very slight modifications to deduce analogous theorems for the problem of Bolza. This will be done in §2 below. By the introduction of new variables, in §3, the parametric problem of Morse and Myers can be immediately reduced to one of the Bolza type. The necessary conditions usually derived from the first variation for the parametric form of the problem turn out to be simple corollaries of the corresponding theorems for the problem of Bolza.

The method used in deducing the theorem of §2 still applies when the function f in the expression (4) is identically zero. The problem is then that of Mayer with variable end-points as formulated by Bliss.§ The theorem deduced by Myers‖ for his modification of this problem is also an immediate consequence of the theorem of §2 below by the method of §3.

2. *The Problem of Bolza.* For a more precise formulation of the problem of Bolza see B_1, §§1 and 8, where the particular case

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* Bolza, loc. cit., p. 433. Bliss, *The problem of Lagrange in the calculus of variations*, American Journal of Mathematics, vol. 52 (1930), condition (d) p. 690. We shall refer to the latter paper in the text by the symbol B_1.

† *MM*, pp. 236, footnote p. 244.

‡ B_1, §8, especially pp. 692, 693.

§ *The problem of Mayer with variable end-points*, Transactions of this Society, vol. 19 (1918), pp. 305–314.

‖ *Adjoint systems in the problem of Mayer under general end conditions*, this Bulletin, vol. 38 (1932), pp. 303–312, Theorem 4 of p. 311.

$G=0$ is investigated, and B_2, §§1, 2. In particular we make the four assumptions B_1, (a), (b), (c), and (d). Furthermore $G(x_1, y_{i1}, x_2, y_{i2})$ is supposed to be of class C' in a neighborhood of the end-points of E_{12}.

The additional term G in the expression I necessitates some slight changes in B_1, §8. Formula $B_1(45)$ becomes

$$\begin{aligned}\lambda_0 I_1(\xi, \eta) = & -\int_{x_1}^{x_2} \lambda_r \zeta_r dx - \lambda_0 f(x_1)\xi_1 - c_i \eta_i(x_1) + \lambda_0 f(x_2)\xi_2 \\ & + \eta_i(x_2) F_{y_i'}(x_2) + \lambda_0 G_{x_1}\xi_1 + \lambda_0 G_{y_{i1}}[\eta_i(x_1) + y'_{i1}\xi_1] \\ & + \lambda_0 G_{x_2}\xi_2 + \lambda_0 G_{y_{i2}}[\eta_i(x_2) + y'_{i2}\xi_2].\end{aligned}$$

The arguments following $B_1(45)$ hold without change and lead to the conclusion that there is a set of constants $\lambda_0, d_1, \cdots, d_p$, not all zero, such that the equation

$$\begin{aligned}0 = & -\int_{x_1}^{x_2} \lambda_r \zeta_r dx + [\lambda_0(-f(x_1) + G_{x_1} + G_{y_{i1}} y'_{i1}) \\ & + d_\mu(\psi_{\mu x_1} + \psi_{\mu y_{i1}} y'_{i1})]\xi_1 \\ & + [\lambda_0(f(x_2) + G_{x_2} + G_{y_{i2}} y'_{i2}) + d_\mu(\psi_{\mu x_2} + \psi_{\mu y_{i2}} y'_{i2})]\xi_2 \\ & + [-c_i + \lambda_0 G_{y_{i1}} + d_\mu \psi_{\mu y_{i1}}]\eta_i(x_1) \\ & + [F_{y_i'}(x_2) + \lambda_0 G_{y_{i2}} + d_\mu \psi_{\mu y_{i2}}]\eta_i(x_2)\end{aligned} \tag{7}$$

holds for every set of admissible variations ξ_1, ξ_2, $\eta_i(x)$ of the arc E_{12}. Choosing the constants c_i so as to make the coefficients of $\eta_i(x_1)$ disappear, we infer, from the fact that ζ_r, ξ_1, ξ_2, $\eta_i(x_2)$ can be chosen arbitrarily for admissible variations, that $\lambda_r \equiv 0$ and also that the matrix

$$\left\| \begin{matrix} -F(x_1) + \lambda_0(G_{x_1} + G_{y_{i1}} y'_{i1}) & -F_{y_i'}(x_1) + \lambda_0 G_{y_{i1}} & & \\ \psi_{\mu x_1} + \psi_{\mu y_{i1}} y'_{i1} & \psi_{\mu y_{i1}} & & \\ & & F(x_2) + \lambda_0(G_{x_2} + G_{y_{i2}} y'_{i2}) & F_{y_i'}(x_2) + \lambda_0 G_{y_{i2}} \\ & & \psi_{\mu x_2} + \psi_{\mu y_{i2}} y'_{i2} & \psi_{\mu y_{i2}} \end{matrix} \right\|$$

is of rank p. Hence also the matrix

$$\left\| \begin{matrix} -F(x_1) + F_{y_i'}(x_1)y_{i1}' + \lambda_0 G_{x_1} & -F_{y_i'}(x_1) + \lambda_0 G_{y_{i1}} & & \\ \psi_{\mu x_1} & \psi_{\mu y_{i1}} & & \\ & & F(x_2) - F_{y_i'}(x_2)y_{i2}' + \lambda_0 G_{x_2} & F_{y_i'}(x_2) + \lambda_0 G_{y_{i2}} \\ & & \psi_{\mu x_2} & \psi_{\mu y_{i2}} \end{matrix} \right\|$$

is of rank p, which is equivalent to the statement that the relation

$$(8) \qquad [(F - F_{y_i'}y_i')dx + F_{y_i'}dy_i]_1^2 + \lambda_0 dG = 0$$

is a consequence of the system of equations

$$(9) \qquad d\psi_\mu \equiv \psi_{\mu x_1}dx_1 + \psi_{\mu y_{i1}}dy_{i1} + \psi_{\mu x_2}dx_2 + \psi_{\mu y_{i2}}dy_{i2} = 0, \qquad (\mu = 1, 2, \cdots, p).$$

The multipliers λ_0, $\lambda_\alpha(x)$ can not all vanish simultaneously at a point $x=a$, since this would imply $B_i \equiv 0$ in equation (17) of paper B_1, and also $v_i(a)=0$, from B_1 (16). Hence B_1 (17) would imply $v_i \equiv 0$; in particular $v_i(x_1)=c_i=0$, and $v_i(x_2)=F_{y_i'}(x_2)=0$. From (7) we then find

$$d_\mu(\psi_{\mu x_1} + \psi_{\mu y_{i1}}y_{i1}') = 0, \qquad d_\mu\psi_{\mu y_{i1}} = 0,$$
$$d_\mu(\psi_{\mu x_2} + \psi_{\mu y_{i2}}y_{i2}') = 0, \qquad d_\mu\psi_{\mu y_{i2}} = 0,$$

from which we infer that $d_\mu=0$, since the rank of the matrix of the coefficients is p. Hence $\lambda_0 = d_\mu = 0$, in contradiction to our previous result. We have thus proved the following theorem.

THEOREM 1. *For every minimizing arc E_{12} for the problem of Bolza there exist constants c_i and a function*

$$(10) \qquad F = \lambda_0 f + \lambda_\alpha(x)\phi_\alpha$$

such that the equations

$$(11) \qquad F_{y_i'} = \int_{x_1}^{x} F_{y_i}dx + c_i, \qquad \phi_\alpha = 0,$$

hold at every point of E_{12}, and furthermore such that the end-points of E_{12} satisfy, besides the equations (3), *the condition that* (8) *holds for every set of differentials dx_1, dy_{i1}, dx_2, dy_{i2} which satisfy the equations* (9). *The first multiplier λ_0 is a constant. The multipliers $\lambda_\alpha(x)$ have definite limits $\lambda_\alpha(x-0)$, $\lambda_\alpha(x+0)$ at every value of x*

on x_1x_2 and these limits are equal and the $\lambda_\alpha(x)$ are continuous except possibly at the values of x defining the corners of E_{12}. The elements of the set λ_0, $\lambda_\alpha(x)$ do not vanish simultaneously at any point of E_{12}.

The form of the expression I to be minimized has no bearing on questions regarding the normality of E_{12}; hence the results of B_1, §9 apply as well for the problem of Bolza.

3. *The Problem of Bolza in the Formulation of Morse and Myers.* In order to derive necessary conditions we suppose that a particular arc E_{12} of class D', whose end-points are given by (5) for $(\alpha)=(0)$, minimizes I in the class of all arcs (1) neighboring E_{12} and satisfying the system (2) and whose end-points are given by (5) for values (α) near the values (0).

The problem of Morse and Myers can be phrased as a problem of Bolza as follows. We adjoin to the set of functions $y_i(x)$ new functions $\alpha_h(x)$ satisfying the differential equations $\alpha_h'(x)=0$. The problem is then that of finding in the class of arcs

$$y_i = y_i(x), \qquad \alpha_h = \alpha_h(x), \qquad (i = 1, \cdots, n; h = 1, \cdots, r; x_1 \leqq x \leqq x_2) \tag{12}$$

satisfying the differential equations and end-conditions

$$\begin{gathered} \phi_\beta(x, y, y') = 0, \qquad \alpha_h' = 0, \qquad (\beta = 1, 2, \cdots, m), \\ \psi_s = x_s - x_s(\alpha_1, \cdots, \alpha_r) = 0, \quad \psi_{is} = y_{is} - y_{is}(\alpha_1, \cdots, \alpha_r) = 0, \end{gathered} \tag{13}$$

one which minimizes the expression I in (6). The particular minimizing arc $E_{12}{}^*$ of the form (12), whose properties are to be studied, has $\alpha_h(x)\equiv 0$, $(h=1, \cdots, r)$. In the end-conditions (13) we think of the constants α_h as the initial values at $x=x_1$ of the functions $\alpha_h(x)$.

All of the assumptions of §2 are satisfied by this problem in the Bolza form. In particular the functional matrix of the end-conditions (13) has its determinant of derivatives with respect to the variables x_1, x_2, y_{i1}, y_{i2} equal to unity. Applying the result of §2 we know that there exist constants c_i, c_{n+h} and a function

$$F = \lambda_0 f + \lambda_\beta(x)\phi_\beta + \lambda_{m+h}(x)\alpha_h'$$

such that the equations

$$F_{y_i'} = \int_{x_1}^{x} F_{y_i} dx + c_i, \qquad F_{\alpha_h'} = \lambda_{m+h} = c_{n+h},$$

hold along E_{12}^*, and furthermore such that at the ends of E_{12}^* the equation

$$(14) \qquad [(F - y_i' F_{y_i'})dx + F_{y_i'} dy_i + \lambda_{m+h} d\alpha_h]_1^2 + \lambda_0 d\theta = 0$$

is a consequence of the equations

$$(15) \qquad dx_s - x_{sh} d\alpha_h = 0, \qquad dy_{is} - y_{ish} d\alpha_h = 0,$$

where x_{sh} and y_{ish} are notations for $\partial x_s/\partial \alpha_h$ and $\partial y_{is}/\partial \alpha_h$ at $(\alpha) = (0)$. The constants λ_{m+h} are all zero since in equation (14) they are coefficients of the differentials $d\alpha_{h2}$ on which no conditions are imposed by the equations (15) which contain only $d\alpha_{h1} = d\alpha_h$. We have therefore proved the following theorem.

THEOREM 2. *With the minimizing arc E_{12} there is associated a set of multipliers*

$$(16) \qquad \lambda_0, \lambda_\beta(x), \qquad (\beta = 1, 2, \cdots, m),$$

and a set of constants $c_1, \cdots, c_n$ such that the equations

$$F_{y_i'} = \int_{x_i}^{x} F_{y_i} dx + c_i$$

are satisfied at every point of E_{12}, where $F = \lambda_0 f + \lambda_\beta \phi_\beta$. Furthermore the equation

$$[(F - F_{y_i'} y_i')dx + F_{y_i'} dy_i]_1^2 + \lambda_0 d\theta = 0$$

is an identity in $d\alpha_h$ when dx_1, dx_2, dy_{i1}, dy_{i2} and $d\theta$ are evaluated for $(\alpha) = (0)$ and expressed in terms of the differentials $d\alpha_h$ by the equations (15). *The elements of* (16) *do not vanish simultaneously at any point of E_{12}.*

According to the definitions of Morse and Myers, and Bliss, the admissible arc E_{12} is normal (*AB*) (see *MM*, §5) if and only if the arc E_{12}^* is normal for the corresponding problem of Bolza (see B_1, §9). Criteria for normality are given in B_1, §9, and *MM*, §5. It can be shown now that Theorem 4 in *MM* follows immediately out of the condition of Bliss concerning the non-vanishing of the determinant $B_1(49)$. From (13), $B_1(44)$, and the equations $\alpha_h' = 0$ we derive for a set of admissible variations along E_{12}^*

$$\begin{aligned} \Psi_s(\xi, \eta, u) &= \xi_s - x_{sh}u_h, \\ \Psi_{is}(\xi, \eta, u) &= \eta_i(x_s) + y'_{is}\xi_s - y_{ish}u_h, \end{aligned} \tag{17}$$

where the constants u_h are the variations of the constant functions $\alpha_h(x)$. From these we find immediately

$$\Psi_{is} = \eta_i(x_s) - \bar{\eta}_{is} + y'_{is}\Psi_s \tag{18}$$

with the notation

$$\bar{\eta}_{is} = (y_{ish} - y'_{is}x_{sh})u_h$$

of Morse and Myers. A necessary and sufficient condition for E_{12}^* to be normal, according to the definition of Bliss, is that there exist $2n+2$ sets of admissible variations $\xi_{s\mu}$, $\eta_{i\mu}(x)$, $u_{h\mu}$, $(\mu=1, \cdots, 2n+2)$, such that the determinant

$$\begin{vmatrix} \Psi_s(\xi_\mu, \eta_\mu, u_\mu) \\ \Psi_{is}(\xi_\mu, \eta_\mu, u_\mu) \end{vmatrix}$$

is different from zero. The first equations (17), and (18), show that this determinant is identical with the determinant *MM* (5.1), with the notations ξ_s in place of γ_s. It follows also that Theorem 5 in *MM* is a corollary of the further results of Bliss in B_1, §9.

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