RESULTS AND PROBLEMS ABOUT n-WEBS OF CURVES IN A PLANE

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If

$$x^* = u(x, y), \qquad y^* = v(x, y)$$

is a topological mapping of the x, y plane, we call the function u *topologically equivalent* to x in this plane. Let us assume n such functions $t_i(x, y)$, $(i=1, 2, \cdots, n)$, in the same simply connected domain D. We have there n *sheaves* of curves $t_i(x, y)=\text{const}$. We call this figure an *n-web* if two curves of different sheaves have not more than one point in common. We suppose the functions $u_{ik}(t_i)$ to be continuous and *strictly monotonic*, so that for all pairs $t_i \neq t_i'$, we have $u_{ik}(t) \neq u_{ik}(t')$.

It seems to be interesting to study n-webs satisfying the condition that there are such functions $u_{ik}(t_i)$ satisfying identically in D the relations

$$\sum_{i=1}^{n} u_{ik}(t_i) = \text{const.}, \quad (k = 1, 2, 3, \cdots, m). \tag{1}$$

We call these equations (1) linearly independent, if the identities

$$\sum_{k=1}^{m} c_k u_{ik}(t_i) = \text{const.} \tag{2}$$

imply for the constants c_k the trivial solution $c_k=0$, $(k=1, 2, 3, \cdots, m)$. The following theorems hold.

THEOREM 1. *A 3-web satisfying one condition* (1), $(n=3, m=1)$, *is topologically equivalent to the tangents of a curve of class* 3 (*irreducible or not*).

This was essentially found by Graf and Sauer† in 1924. Howe and I‡, in 1932, proved the following theorem.

† H. Graf and R. Sauer, Münchener Berichte, 1924; W. Blaschke and G. Howe, Hamburg Abhandlungen, vol. 9 (1932); W. Blaschke, Tôhoku Mathematical Journal, 1933.

‡ W. Blaschke and G. Howe, Hamburg Abhandlungen, vol. 9 (1932).

THEOREM 2. *A straight lined n-web satisfying (at least) one condition* (1) *is necessarily equivalent to the tangents of a curve of class n* ($n\geqq 3$).

Our result contains Theorem 1 as a special case, because a 3-web satisfying the equation

$$u_1 + u_2 + u_3 = \text{const.}$$

is equivalent to a special straight lined 3-web (*hexagonal web*), as we see if we assume u_1 and u_2 as parallel coordinates.

Howe observed that the following Theorem 3 is equivalent to S. Lie's results about the surfaces, which are *translation surfaces* in different ways.

THEOREM 3. *A* 4-*web satisfying* 3 *linearly independent relations* ($n=4$, $m=3$) *is equivalent to the tangents of a curve of class* 4.

A geometric interpretation of one condition (1) for a 4-web has been given by Bose and myself.† Bol‡ discovered a short time ago the following result.

THEOREM 4. *The maximum number m of linearly independent relations* (1) *for an n-web is*

$$m = \frac{(n-1)(n-2)}{2}\,. \tag{3}$$

Almost equivalent to a theorem of Reidemeister§ are the following.

THEOREM 5. *A* 4-*web satisfying* 3 *linearly independent relations* (1) *with* $u_{ii}=0$ *is equivalent to* 4 *pencils of straight lines, no* 3 *of the* 4 *vertices on a straight line.*

THEOREM 6. *A* 4-*web satisfying* 3 *relations* (1) *with* $u_{ii}=0$, ($i=1, 2, 3$), *only two of them linearly independent, admits a continuous one-parameter group, the* $t_4=\text{const.}$ *being paths.*

But between the proofs of these theorems there is an essential difference Only Theorem 1 and the greater part of Theorems 5 and 6 are proved without any further restrictions for the functions t_i, u_{ik}. The proofs already known for Theorems 2–6

† W. Blaschke and R. C. Bose, Indian Physico-Mathematical Journal, vol 3 (1932), p. 99.

‡ G. Bol, this Bulletin, vol. 38 (1932), pp. 855–857.

§ K. Reidemeister, Mathematische Zeitschrift, vol. 29 (1928).

contain regularity restrictions. Therefore the first problem to be solved is the following one.

PROBLEM A. *Do the Theorems* 2–5 *remain valid without further regularity restrictions?*

Another question unsolved as far as I know is the following one.

PROBLEM B. *To extend our Theorem* 3 *to n-webs.*

These problems seem to be interesting because, for example, they contain a kind of real geometrical interpretation of Abel's theorem on algebraic curves.

Finally a few words about more dimensions. The questions about *webs of surfaces*

$$S_i(x, y, z) = \text{const.}$$

in a 3-space can partially be reduced to our theorems on curve-webs in a 2-space. But if we consider *sheaves of curves*

$$s_i(x, y, z) = \text{const.}, \; t_i(x, y, z) = \text{const.}$$

in a 3-space we may ask, for example, the following question.

PROBLEM C. *How many essentially different relations*

$$u_1(s_1, t_1) + u_2(s_2, t_2) + u_3(s_3, t_3) = \text{const.}$$

can exist for a 3-web of curves

$$s_i, t_i = \text{const.}, \qquad (i = 1, 2, 3),$$

in a 3-space?

This seems to me to be one of the most promising fields of geometric research.

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