THE POISSON INTEGRAL FOR FUNCTIONS WITH POSITIVE REAL PART*

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The aim of this paper is to derive a Poisson integral representation valid for all functions $g(\lambda)$ which are regular in the right λ-half-plane, and for which†

$$\mathfrak{R}g(\lambda) \geqq 0$$

in that half-plane, and in particular for such functions $g(\lambda)$ which are real for real values of λ. This latter class of functions as well as their Poisson integral representations play a fundamental role in the theory of alternating current networks.‡ The resulting Poisson integral representation (equation (10)) is a very simple one and closely connected with the theory of Stieltjes' continued fractions. These facts seem to justify the elementary derivation presented here, though the equivalent Poisson integral for the unit circle is well known.

Herglotz§ has proved¶ the theorem: *Every function $f(z)$ regular in the interior of the unit circle with real part not negative (and only such functions) can be represented as*

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\alpha} + z}{e^{i\alpha} - z}\, d\mu(\alpha) + k, \tag{1}$$

the integral being taken in the Stieltjes sense, μ being a non-decreasing bounded function, k a pure imaginary constant.

Taking the real part we obtain the Poisson integral for any

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† $\mathfrak{R}$ means "real part of."

‡ W. Cauer, Jahresberichte der deutschen Mathematiker Vereinigung, 1929; Mathematische Annalen, vol. 105 (1931); vol. 106 (1932); O. Brune, Journal of Mathematics and Physics, Mass. Inst. of Tech., 1931.

§ G. Herglotz, *Über Potenzreihen mit positivem reellen Teil im Einheitskreis*, Leipziger Berichte, vol. 63 (1911). Herglotz also gives an explicit expression for $\mu(\phi)$ in terms of the coefficients of the power series for $f(z)$.

¶ Another proof follows by theorems of Helly, Wiener Sitzungsberichte, vol. 121 (IIa) (1912), p. 283 and 288. See also Evans, *The Logarithmic Potential*, 1927, p. 46; Bray, Annals of Mathematics, (2), vol. 20 (1919), p. 180, Theorem 3; T. H. Hildebrandt, this Bulletin, vol. 28 (1922), pp. 53–58.

positive harmonic function in the unit circle

$$(2)\quad u(r, \phi) = \Re f(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos(\phi - \alpha) + r^2} d\mu(\alpha),$$

where $z = re^{i\phi}$.

The following derivation of the Poisson integral for the right λ-half-plane as mentioned in the beginning depends on formula (1). Since any function of bounded variation can be represented as a difference of two non-decreasing bounded functions, in the sequel $\mu(\phi)$ may be supposed to be a non-decreasing function. For simplicity we redefine $\mu(\phi)$ at all interior points of discontinuity (if necessary) in such a way, that $\frac{1}{2}[\mu(\phi+0)+\mu(\phi-0)] = \mu(\phi)$. The substitution

$$(3)\qquad z = \frac{\lambda - 1}{\lambda + 1}$$

transforms the interior of the unit circle into the right λ-half-plane, real λ corresponding to real z. Similarly

$$e^{i\alpha} = \frac{iy - 1}{iy + 1},$$

$$y = \operatorname{ctn}\frac{\alpha}{2},$$

where the interval $0 < \alpha < 2\pi$ corresponds to $\infty > y > -\infty$, gives a corresponding transformation of the variable of integration.

If we write

$$(4)\qquad -\frac{1}{2\pi}\mu(\alpha) = \nu(y),$$

then the transformed Poisson integral becomes

$$f(z) = g(\lambda) = -\int_{\operatorname{ctn}\epsilon/2}^{-\operatorname{ctn}\epsilon/2} \frac{(iy-1)(\lambda+1) + (iy+1)(\lambda-1)}{(iy-1)(\lambda+1) - (iy+1)(\lambda-1)} d\nu(y)$$
$$+ \frac{1}{2\pi}\int_0^{\epsilon} \frac{e^{i\alpha} + z}{e^{i\alpha} - z} d\mu(\alpha) + \frac{1}{2\pi}\int_{2\pi-\epsilon}^{2\pi} \frac{e^{i\alpha} + z}{e^{i\alpha} - z} d\mu(\alpha) + k.$$

Since $d\alpha = -2dy/(1+y^2)$, $\nu(y)$ is a non-decreasing function.

Considering the possible jump of $\mu(\phi)$ at $\phi=0$ and $\phi=2\pi$, and the relation

$$\frac{1+z}{1-z}=\lambda,$$

we obtain, by passing to the limit as $\epsilon\to 0$, the following result.

THEOREM 1. *Every function $g(\lambda)$, regular in the right λ-half-plane with $\Re g(\lambda)\geqq 0$ there, can be represented as*

$$g(\lambda)=\int_{-\infty}^{+\infty}\frac{iy\lambda-1}{iy-\lambda}\,d\nu(y)+C\lambda+k, \tag{5}$$

where ν is a non-decreasing bounded function, C a not negative and k a pure imaginary constant.

In connection with Stieltjes' moment problem Rolf Nevanlinna* has studied functions $h(x)$ *regular in the upper x-half-plane having there a non-positive imaginary part.* To obtain a Poisson integral formula for such functions (5) must be transformed by

$$i\lambda = x,$$
$$-ig(\lambda)=h(x),$$

so that†

$$h(x)=\int_{-\infty}^{\infty}\frac{1-xy}{x+y}\,d\nu(y)-Cx+l, \tag{6}$$

where ν is a non-decreasing bounded function, C a not negative

* Rolf Nevanlinna, *Asymptotische Entwicklungen beschränkter Funktionen und das Stieltjes'sche Momentenproblem,* Suomalaisen Tiedeakatemian Kustantama, 1922.

† The more direct but not shorter derivation of R. Nevanlinna (loc. cit, p. 44) gives the result

$$h(x)=\int_{-\infty}^{\infty}\frac{d\chi(u)}{x-u},$$

χ non-decreasing, for all functions $h(x)$ allowing the asymptotic development $c_1/x+c_2/x^2+c_3/x^3+\cdots$ for $\epsilon<\operatorname{arc} x<\pi-\epsilon$, where ϵ is an arbitrary small positive number. This supposition requires the existence of $\int_{-\infty}^{\infty}u^n d\chi(u)$ for each positive n. But, according to (6), Nevanlinna's formula holds if we suppose only that $\lim_{r=\infty}\left|re^{i\phi}h(re^{i\phi})\right|<K$ for some positive constant K and some ϕ of the interval $0<\phi<\pi$ and if $\int_a^{\infty}d\chi(u)/u$ exists for $a>0$. Then $\chi(u)=-\int(1+u^2)d\nu(-u)$.

and l a real constant.

Now $g(\lambda)$ of (5) may be further restricted to be real for real values of λ or, what amounts to the same thing, $f(z)$ real for real z and, by the Schwarz reflection principle,

$$f(\bar{z}) = \bar{f}(z),$$

where the dash indicates the conjugate complex value.

This implies

$$u(r, 2\pi - \phi) = u(r, \phi), \; u = \Re f(z), \; z = re^{i\phi}.$$

From the formula*

$$\lim_{r=1} \int_{\phi_1}^{\phi_2} u(r, \alpha) d\alpha = \mu(\phi_2) - \mu(\phi_1), \quad (0 < \phi_1 < \phi_2 < 2\pi), \tag{7}$$

it follows that

$$\mu(\phi_2) - \mu(\phi_1) = \mu(2\pi - \phi_1) - \mu(2\pi - \phi_2),$$

and therefore by (4),

$$\nu(-y) = -\nu(y) + \text{const.} \tag{8}$$

Then the integral of (5) can be written

$$\int_0^\infty \left(\frac{iy\lambda - 1}{iy - \lambda} + \frac{-iy\lambda - 1}{-iy - \lambda} \right) d\nu(y) = 2\lambda \int_0^\infty \frac{(1 + y^2) d\nu(y)}{\lambda^2 + y^2}.$$

Since this is real for real λ, $g(\lambda)$ becomes real for real λ when and only when $k=0$. On defining the non-decreasing function $\psi(x)$ by

$$\psi(x) = 2 \int_0^{\sqrt{x}} (1 + y^2) d\nu(y) \tag{9}$$

we get the result:

THEOREM 2. *Any function* $g(\lambda)$ *which is regular in the right* λ*-half-plane and such that* $\Re g(\lambda) \geqq 0$ *there and* $g(\lambda)$ *is real for real* λ, *can be represented as*

$$g(\lambda) = \lambda \left[C + \int_0^\infty \frac{d\psi(x)}{\lambda^2 + x} \right], \tag{10}$$

* See, for instance, G. C. Evans, loc. cit., p. 37, equation (14).

where C is a non-negative constant, ψ a non-decreasing function and the integral is to be taken in the Stieltjes' sense. Conversely if integral (10) *exists, it represents a function of the class specified.*

The converse part of Theorem 2 can be verified immediately. ψ is not necessarily bounded, but it is obvious that the integral (10) exists if and only if

$$\lim_{b=\infty} \int_a^b \frac{d\psi(x)}{x}$$

exists for $a>0$. Since

$$\int_a^b \frac{d\psi(x)}{x} = \frac{1}{\xi}[\psi(b) - \psi(a)], \qquad (a \leqq \xi \leqq b),$$

a necessary condition is that $\psi(x)/x$ be bounded for $x>a>0$. A sufficient condition for the existence of (10) is that $\psi(x) < Ax^{\alpha}$, where A and α are positive constants and $\alpha<1$.

By (7), ψ is connected with the real part of the integral function of g. The relation is

$$\psi(x) = \frac{2}{\pi} \lim_{\epsilon=0} \Re \int_0^{\sqrt{x}} g(\epsilon + iy)dy.*$$

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* The direct proof of Perron in *Die Lehre von den Kettenbrüchen*, 2d edition, 1929, p. 372–374 is applicable to our case, because Perron's restriction that $\int_0^\infty d\psi(x)$ exists is superfluous for his proof, if a slight modification is made.